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RESEARCH ARTICLE

Map composition generalized to coherent collections of maps

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Abstract Relation algebras give rise to partial algebras on maps, which are generalized to partial algebras on polymaps while preserving the properties of relation union and composition. A polymap is defined as a map with every point in the domain associated with a special set of maps. Polymaps can be represented as small subcategories of **Set***, the category of pointed sets. Map composition and the counterpart of relation union for maps are generalized to polymap composition and sum. Algebraic structures and categories of polymaps are investigated. Polymaps present the unique perspective of an algebra that can retain many of its properties when its elements (maps) are augmented with collections of other elements.

Keywords Relation algebra, partial algebra, composition
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1 Introduction

This paper considers partial algebras on maps (similar to those defined in [15]), constructed in analogy to relation algebras [5,12,16]. The sum of two maps $f_0\colon X_0 \to Y_0$ and $f_1\colon X_1 \to Y_1$ is defined if the union of their graphs is a functional binary relation

$$R \subseteq (X_0 \cup X_1) \times (Y_0 \cup Y_1),$$

in which case the sum is the map $(f_0 + f_1)\colon X_0 \cup X_1 \to Y_0 \cup Y_1$ with graph R. This is defined similarly to the union operation + from [10]. Map composition corresponds to relation composition from relation algebra. These definitions give rise to the following properties, some familiar from relation algebra.

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- Map composition is associative when defined.
- The identity maps serve as map composition identities.
- Map summation is idempotent, commutative, and associative.
- Map composition right-distributes over map summation.
- The empty map is a summation identity.

We generalize this partial algebra, while preserving the above properties, by augmenting any map $h\colon X \to Y$ with additional structure, that is, associating each $x \in X$ with a set of maps $H \subseteq Y^X$ such that

$$h'(x) = h(x), \quad \forall\, h' \in H. \tag{1.1}$$

Such a structure is called a *polymap* (Definition 2.1). Polymaps and the operations between them are shown to be significant extensions of maps, and their operations, that exhibit several useful properties. Thus, we have algebras that preserve many of their properties when we augment their elements, maps, with collections of other elements. This is a potentially novel insight with regards to algebra.

Polymaps are motivated from an abstract generalization of the technique of *grafting* in origami as applied to substrates P such as topological surfaces and even discrete substrates like abstract graphs. Origami is generally defined as a continuous path isometry $f\colon P \to \mathbb{R}^3$ that does not cause self-intersection, where P is the initial sheet of paper. Grafting is an established origami design technique, which involves folding a sheet P_0 via an origami $f_0\colon P_0 \to \mathbb{R}^3$ to some configuration $f_0(P_0)$, after which $f_0(P_0)$ is treated as a fresh sheet of paper P_1 and another origami $f_1\colon P_1 \to \mathbb{R}^3$ is folded on it. This process is a potent origami design technique [3] using only one sheet P_0.

This can be stated more precisely by considering Belcastro and Hull's definition of origami as a piecewise rigid map [1]. That is, each origami $f\colon P \to \mathbb{R}^3$ corresponds to a family $\{(P_i, A_i)\}_{i\in I}$ such that $\{P_i\}_{i\in I}$ is a partition of P, and each $A_i\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a proper rigid transformation such that $f|_{P_i} = A_i|_{P_i}$ for all $i \in I$ (corresponding to (1.1)).

As a polymap, each $x \in P_i$ is assigned the set $\{A_i\}$. The result of grafting an origami $g\colon f(P) \to \mathbb{R}^3$ (corresponding to $\{(Q_j, B_j)\}_{j\in J}$) after f is an origami $P \to \mathbb{R}^3$, for which every $x \in P_i$ is assigned the set $\{B_j \circ A_i\}$ if $f(x) \in Q_j$. This forms the basis of Definition 2.6 below for polymap composition.

Section 2 defines polymaps and develops the basic properties of composition on polymaps, as well as inverse polymaps, in analogy to maps. Section 3 generalizes the category **Set** to the categories **SetP**, **SetTP**, and **SetM**, whose objects are sets and hom-sets consist of polymaps. Definitions from category theory follow [11] and [17]. Group-like algebraic structures and semilattices of polymaps are constructed based on the polymap sum in Section 4, including an idempotent involution semiring that has no nontrivial natural analogue in terms of maps. Polymaps are also represented as small subcategories of the category of pointed sets, **Set***, in Section 5.

2 Polymaps and their operations

Definition 2.1 (Polymap) For any sets X and Y and a map $f'\colon X' \to Y$, where $X' \subseteq X$, a *polymap* from X via f' to Y is a triplet $\mathsf{f} = (X, \phi, Y)$, where $\phi\colon X \to \mathscr{P}(Y^X)$ is a map such that $\phi(x) \neq \emptyset \iff x \in X'$ and $f(x) = f'(x)$ for all $f \in \phi(x)$ (corresponding to (1.1)). Such a polymap is denoted as $\mathsf{f}\colon X \Rrightarrow Y$. Let $\operatorname{dom}\mathsf{f} = X'$, $\langle \mathsf{f} \rangle = \phi$, and $\operatorname{im}\mathsf{f} = f'(X')$. Clearly, f' is unique to f so it is denoted as $\overleftarrow{\mathsf{f}}$. If $|\phi(x)| \leqslant 1$ for all $x \in X$, then f is called a *monomap*. If $X' = X$, then f is called *total*.

Example 2.2 Given any nonempty set of maps $F \subseteq Y^X$, the *constant polymap* via F is defined as $\mathsf{C}_F\colon X \Rrightarrow Y$, where

$$\langle \mathsf{C}_F \rangle(x) = \begin{cases} F, & x \in \operatorname{Eq} F, \\ \emptyset, & x \notin \operatorname{Eq} F, \end{cases}$$

and $\operatorname{Eq} F = \{x \in X \mid \forall\, f_0, f_1 \in F\colon f_0(x) = f_1(x)\}$ is the equalizer of F.

Define a partial order $\leqslant$ on the class of all polymaps, where $\mathsf{f}_0 \leqslant \mathsf{f}_1$ (f_0 is a *subpolymap* of f_1) if and only if $\operatorname{dom}\mathsf{f}_0 \subseteq \operatorname{dom}\mathsf{f}_1$ and $\langle \mathsf{f}_0 \rangle(x) \subseteq \langle \mathsf{f}_1 \rangle(x)$ for all $x \in \operatorname{dom}\mathsf{f}$. If $\mathsf{f}_0 \leqslant \mathsf{f}_1$, and $\operatorname{dom}\mathsf{f}_0 = \operatorname{dom}\mathsf{f}_1$ (resp. $\langle \mathsf{f}_0 \rangle(x) = \langle \mathsf{f}_1 \rangle(x)$ for all $x \in \operatorname{dom}\mathsf{f}_0$), call f_0 a *wide* (resp. *full*) subpolymap of f_1. This nomenclature is motivated by the interpretation of polymaps as categories in Section 5, in particular Lemma 5.2.

Example 2.3 Given any singleton set $\{y\}$, the only total polymap from a given set X to $\{y\}$ is the total monomap $\mathsf{C}_{\{f\}}$, where $f\colon X \to \{y\}$ is a constant map. The polymaps $\mathsf{f}\colon X \Rrightarrow \{y\}$ are precisely the subpolymaps of $\mathsf{C}_{\{f\}}$.

Example 2.4 Given some $X' \subseteq X$, the *inclusion polymap* on X from X' is defined as $1^X_{X'}\colon X \Rrightarrow X$, where

$$\langle 1^X_{X'} \rangle(x) = \begin{cases} \{\mathrm{id}\}, & x \in X', \\ \emptyset, & x \notin X'. \end{cases}$$

The term "inclusion polymap" is motivated by the observation that $\overleftarrow{1^X_{X'}}$ is the inclusion map $X' \hookrightarrow X$.

Example 2.5 The only polymap from $\emptyset$ to any given set Y is the polymap $(\emptyset, p, Y)$, where $p\colon \emptyset \to \mathscr{P}(Y^X)$ is an empty map. Such a polymap is vacuously a total monomap.

2.1 Polymap composition

A composition operation on polymaps is defined in direct analogy to map composition. Let $\mathsf{f}\colon X \Rrightarrow Y$, $\mathsf{g}\colon Y \Rrightarrow Z$, and $\mathsf{h}\colon Z \Rrightarrow W$ be polymaps. Given any $F \subseteq Y^X$ and $G \subseteq Z^Y$, let $G \circledcirc F = \{g \circ f \mid g \in G,\ f \in F\}$. Note that $\circledcirc$ is associative and $F \circledcirc \emptyset = \emptyset = \emptyset \circledcirc F$.

Definition 2.6 Define the *polymap composite of* g *after* f as $\mathsf{g} \bullet \mathsf{f}\colon X \Rrightarrow Z$, where

$$\langle \mathsf{g} \bullet \mathsf{f} \rangle(x) = \begin{cases} \langle \mathsf{g} \rangle(\overleftarrow{\mathsf{f}}(x)) \circledcirc \langle \mathsf{f} \rangle(x), & x \in \operatorname{dom} \mathsf{f}, \\ \emptyset, & x \notin \operatorname{dom} \mathsf{f}. \end{cases}$$

It can be verified easily that every polymap composite is a polymap.

The analogy with map composition is shown in the following lemma.

Lemma 2.7 *Each* $x \in \operatorname{dom}(\mathsf{g} \bullet \mathsf{f}) = \overleftarrow{\mathsf{f}}^{-1}(\operatorname{dom} \mathsf{g})$ *satisfies* $\overleftarrow{\mathsf{g} \bullet \mathsf{f}}(x) = \overleftarrow{\mathsf{g}}(\overleftarrow{\mathsf{f}}(x))$.

Proof From the definition of polymap composition, $\operatorname{dom}(\mathsf{g} \bullet \mathsf{f}) \subseteq \operatorname{dom} \mathsf{f}$. Each $x \in \operatorname{dom} \mathsf{f}$ satisfies $\langle \mathsf{f} \rangle(x) \neq \emptyset$ and

$$\begin{aligned} x \in \operatorname{dom}(\mathsf{g} \bullet \mathsf{f}) &\iff \langle \mathsf{g} \bullet \mathsf{f} \rangle(x) \neq \emptyset \\ &\iff \langle \mathsf{g} \rangle(\overleftarrow{\mathsf{f}}(x)) \neq \emptyset \\ &\iff \overleftarrow{\mathsf{f}}(x) \in \operatorname{dom} \mathsf{g} \\ &\iff x \in \overleftarrow{\mathsf{f}}^{-1}(\operatorname{dom} \mathsf{g}). \end{aligned}$$

Suppose that $x \in \overleftarrow{\mathsf{f}}^{-1}(\operatorname{dom} \mathsf{g})$. Choose any $f \in \langle \mathsf{f} \rangle(x)$ and $g \in \langle \mathsf{g} \rangle(\overleftarrow{\mathsf{f}}(x))$; by the definition of polymap composition, $g \circ f \in \langle \mathsf{g} \bullet \mathsf{f} \rangle(x)$. By the definition of a polymap,

$$\overleftarrow{\mathsf{g} \bullet \mathsf{f}}(x) = g(f(x)) = \overleftarrow{\mathsf{g}}(\overleftarrow{\mathsf{f}}(x)).$$

□

Lemma 2.8 *Polymap composition is associative. That is,*

$$\mathsf{h} \bullet (\mathsf{g} \bullet \mathsf{f}) = (\mathsf{h} \bullet \mathsf{g}) \bullet \mathsf{f}.$$

Proof It is first demonstrated that

$$\overleftarrow{\mathsf{g} \bullet \mathsf{f}}^{-1}(\operatorname{dom} \mathsf{h}) = \overleftarrow{\mathsf{f}}^{-1}(\overleftarrow{\mathsf{g}}^{-1}(\operatorname{dom} \mathsf{h})).$$

($\supseteq$) For all $x_0 \in \overleftarrow{\mathsf{f}}^{-1}(\overleftarrow{\mathsf{g}}^{-1}(\operatorname{dom} \mathsf{h})) \subseteq \overleftarrow{\mathsf{f}}^{-1}(\operatorname{dom} \mathsf{g})$, Lemma 2.7 gives

$$\overleftarrow{\mathsf{g} \bullet \mathsf{f}}(x_0) = \overleftarrow{\mathsf{g}}(\overleftarrow{\mathsf{f}}(x_0)) \in \operatorname{dom} \mathsf{h}.$$

($\subseteq$) Each $x_1 \in \overleftarrow{\mathsf{g} \bullet \mathsf{f}}^{-1}(\operatorname{dom} \mathsf{h}) \subseteq \operatorname{dom}(\mathsf{g} \bullet \mathsf{h})$ satisfies $\langle \mathsf{g} \bullet \mathsf{f} \rangle(x_1) \neq \emptyset$, and thus, $x_1 \in \overleftarrow{\mathsf{f}}^{-1}(\operatorname{dom} \mathsf{g})$. By Lemma 2.7,

$$\overleftarrow{\mathsf{g}}(\overleftarrow{\mathsf{f}}(x_1)) = \overleftarrow{\mathsf{g} \bullet \mathsf{f}}(x_1) \in \operatorname{dom} \mathsf{h}.$$

Thus,

$$\operatorname{dom}(\mathsf{h} \bullet (\mathsf{g} \bullet \mathsf{f})) = \overleftarrow{\mathsf{f}}^{-1}(\overleftarrow{\mathsf{g}}^{-1}(\operatorname{dom} \mathsf{h})) = \operatorname{dom}((\mathsf{h} \bullet \mathsf{g}) \bullet \mathsf{f}).$$

For any $x \in \overleftarrow{\mathsf{f}}^{-1}(\overleftarrow{\mathsf{g}}^{-1}(\operatorname{dom}\mathsf{h}))$, we have

$$\begin{aligned}
\langle \mathsf{h} \bullet (\mathsf{g} \bullet \mathsf{f})\rangle(x) &= \langle \mathsf{h}\rangle(\overleftarrow{\mathsf{g} \bullet \mathsf{f}}(x)) \circledcirc \langle \mathsf{g} \bullet \mathsf{f}\rangle(x) \\
\text{(by Lemma 2.7)} \qquad &= \langle \mathsf{h}\rangle(\overleftarrow{\mathsf{g}}(\overleftarrow{\mathsf{f}}(x))) \circledcirc \langle \mathsf{g}\rangle(\overleftarrow{\mathsf{f}}(x)) \circledcirc \langle \mathsf{f}\rangle(x) \\
&= \langle \mathsf{h} \bullet \mathsf{g}\rangle(\overleftarrow{\mathsf{f}}(x)) \circledcirc \langle \mathsf{f}\rangle(x) \\
&= \langle (\mathsf{h} \bullet \mathsf{g}) \bullet \mathsf{f}\rangle(x).
\end{aligned}$$

□

Remark 2.9 Given polymaps $\mathsf{f}\colon X \Rrightarrow Y$ and $\mathsf{g}\colon Y \Rrightarrow Z$, it may be possible to define a generalized polymap composite $\mathsf{h}\colon X \Rrightarrow Z$ such that $\langle \mathsf{h}\rangle(x)$ has a useful nonempty definition even if $x \notin \overleftarrow{\mathsf{f}}^{-1}(\operatorname{dom}\mathsf{g})$, as opposed to Definition 2.6. This can be achieved if $\overleftarrow{\mathsf{f}}(x)$ "derives" its transformations under g from its surroundings. That is, if C is the component of $\operatorname{im}\mathsf{f} \setminus \operatorname{dom}\mathsf{g}$ containing $\overleftarrow{\mathsf{f}}(x)$, and $\langle \mathsf{g}\rangle$ has constant value G over the boundary of C, then $\langle \mathsf{h}\rangle(x)$ may be defined as $G \circledcirc \langle \mathsf{f}\rangle(x)$ instead of $\langle \mathsf{g}\rangle(\overleftarrow{\mathsf{f}}(x)) \circledcirc \langle \mathsf{f}\rangle(x)$.

However, it is likely that a topological context is unnecessary, thus the bare notion of "connectedness" may be utilized, as described using the connective spaces of Muscat and Buhagiar [13] and connectivity spaces of Dugowson [7] (or the objects of Börger's category *Zus* [2]).

2.2 Inverse polymap

f is called *polyinjective* (resp. *polysurjective*) if every member of $\bigcup_{x\in X}\langle \mathsf{f}\rangle(x)$ is injective (resp. surjective). If f is both polyinjective and polysurjective, then it is called *polybijective*. If f is polybijective and $\overleftarrow{\mathsf{f}}$ is injective, then it is called *invertible*, in which case its *inverse* is defined as $\mathsf{f}^{-1}\colon Y \Rrightarrow X$, where

$$\langle \mathsf{f}^{-1}\rangle(y) = \begin{cases} \{f^{-1} \mid f \in \langle \mathsf{f}\rangle(x)\}, & \exists\, x \in X\colon \overleftarrow{\mathsf{f}}(x) = y, \\ \emptyset, & \nexists\, x \in X\colon \overleftarrow{\mathsf{f}}(x) = y. \end{cases}$$

Inverting a polymap (finding its inverse) is clearly an involutory operation; the inverse polymap is clearly invertible. Other properties of this involution, especially in relation to semigroup operations [4,8], are developed in Subsection 4.2. The polymap inverse is not entirely an analogue of map inverse (consider Theorem 3.5); instead, its properties are more closely related to those of relation inverse. For example, the following lemma corresponds to the switching of relation image and preimage under the relation inverse.

Lemma 2.10 *If* f *is invertible, then* $\operatorname{dom}(\mathsf{f}^{-1}) = \operatorname{im}\mathsf{f}$.

Lemma 2.11 *If* f *is invertible, then the graph of* $\overleftarrow{\mathsf{f}^{-1}}$ *is the relation inverse of the graph of* $\overleftarrow{\mathsf{f}}$.

Proof For each (y_0, x_0) in the graph of $\overleftarrow{\mathsf{f}^{-1}}$, pick some $e \in \langle \mathsf{f}^{-1}\rangle(y_0)$. We have $\overleftarrow{\mathsf{f}^{-1}}(y_0) = e(y_0) = x_0$. The definition of inverse polymap guarantees the existence of some $x_0' \in X$ such that $\overleftarrow{\mathsf{f}}(x_0') = y_0$ and $e^{-1} \in \langle \mathsf{f} \rangle (x_0')$. That is, $e^{-1}(x_0') = y_0$, and thus, $x_0' = e(y_0) = x_0$. Therefore, $\overleftarrow{\mathsf{f}}(x_0) = y_0$ and (x_0, y_0) lies in the graph of $\overleftarrow{\mathsf{f}}$.

Conversely, for each (x_1, y_1) in the graph of $\overleftarrow{\mathsf{f}}$, pick some $f \in \langle \mathsf{f} \rangle (x_1)$. We have $\overleftarrow{\mathsf{f}}(x_1) = f(x_1) = y_1$. The definition of inverse polymap guarantees that $f^{-1} \in \langle \mathsf{f}^{-1} \rangle (y_1)$, that is, $\overleftarrow{\mathsf{f}^{-1}}(y_1) = f^{-1}(y_1) = x_1$. Therefore, the graph of $\overleftarrow{\mathsf{f}^{-1}}$ contains (y_1, x_1).

To conclude, the graph of $\overleftarrow{\mathsf{f}^{-1}}$ is the relation inverse of the graph of $\overleftarrow{\mathsf{f}}$. □

Lemma 2.12 *Suppose that* f *is invertible. Then the following statements are equivalent*:

(i) f^{-1} *is total,*

(ii) $\overleftarrow{\mathsf{f}}$ *is surjective,*

(iii) $\overleftarrow{\mathsf{f}^{-1}} = \overleftarrow{\mathsf{f}}^{-1}$.

Proof (i) $\iff$ (ii) By Lemma 2.10, $\operatorname{dom} \mathsf{f}^{-1} = Y$ if and only if the image of $\overleftarrow{\mathsf{f}}$ is $\operatorname{im} \mathsf{f} = Y$.

(ii) $\implies$ (iii) If $\overleftarrow{\mathsf{f}}$ is surjective, then it is also bijective. Lemma 2.11 guarantees that $\overleftarrow{\mathsf{f}^{-1}}$ is bijective as well because it is the relation inverse of the graph of $\overleftarrow{\mathsf{f}}$. Together these imply that $\overleftarrow{\mathsf{f}^{-1}} = \overleftarrow{\mathsf{f}}^{-1}$.

(iii) $\implies$ (ii) If $\overleftarrow{\mathsf{f}}$ is invertible, it must be surjective. □

Remark 2.13 The similarities between polymaps and relations described above, the origins of polymaps from relation algebra, as well as the fact that polymaps need not be total (corresponding to left-totality of relations), suggest a possible generalization of polymaps to *polyrelations*. A polyrelation could be defined as a relation $R \subseteq X \times Y$ with every $(x, y) \in R$ being assigned a relation $S \subseteq X \times Y$ containing (x, y). Polymaps are effectively polyrelations in which R and every S are functional.

3 Categories of polymaps

The properties of polymaps and their composition allow the formation of a category **SetP** (resp. **SetM**, **SetTP**) on the class of sets, whose morphisms from X to Y are polymaps (resp. monomaps, total polymaps) from X to Y. The morphisms are composed via polymap composition, and the identity on X is 1_X^X. **SetM** and **SetTP** are wide subcategories of **SetP**. There is a full functor S: **SetTP** $\to$ **Set** that maps sets to themselves and every total polymap f to $\overleftarrow{\mathsf{f}}$. Consider the map P: **Set** $\to$ **SetP** which maps sets to themselves and every morphism f to $\mathsf{C}_{\{f\}}$. The functor P with codomain restricted to **SetP** (resp. **SetM**, **SetTP**) is an *embedding* [11] from **Set** to **SetP** (resp. **SetM**, **SetTP**).

Define an equivalence relation $\approx$ on polymaps, where given $\mathsf{f}_0\colon X_0 \Rrightarrow Y_0$ and $\mathsf{f}_1\colon X_1 \Rrightarrow Y_1$, $\mathsf{f}_0 \approx \mathsf{f}_1$ if and only if $(X_0, Y_0) = (X_1, Y_1)$ and $\overleftarrow{\mathsf{f}_0} = \overleftarrow{\mathsf{f}_1}$. Recall the definition of a *congruence*, which allows an equivalence relation on morphisms to induce a *quotient category* [11].

Lemma 3.1 *$\approx$ is a congruence on any subcategory of* **SetP**.

Proof Consider any subcategory $\mathbf{C}$ of **SetP**, as well as some

$$\mathsf{f}_0 \approx \mathsf{f}_1 \in \hom_{\mathbf{C}}(X, Y), \quad \mathsf{g}_0 \approx \mathsf{g}_1 \in \hom_{\mathbf{C}}(Y, Z).$$

It suffices to show that $\mathsf{g}_0 \bullet \mathsf{f}_0 \approx \mathsf{g}_1 \bullet \mathsf{f}_1$. Since $\overleftarrow{\mathsf{f}_0} = \overleftarrow{\mathsf{f}_1}$ and $\overleftarrow{\mathsf{g}_0} = \overleftarrow{\mathsf{g}_1}$, note that

$$\operatorname{dom}(\mathsf{g}_0 \bullet \mathsf{f}_0) = \overleftarrow{\mathsf{f}_0}^{-1}(\operatorname{dom} \mathsf{g}_0) = \overleftarrow{\mathsf{f}_1}^{-1}(\operatorname{dom} \mathsf{g}_1) = \operatorname{dom}(\mathsf{g}_1 \bullet \mathsf{f}_1).$$

By Lemma 2.7,

$$\overleftarrow{\mathsf{g}_0 \bullet \mathsf{f}_0}(x) = \overleftarrow{\mathsf{g}_0}(\overleftarrow{\mathsf{f}_0}(x)) = \overleftarrow{\mathsf{g}_1}(\overleftarrow{\mathsf{f}_1}(x)) = \overleftarrow{\mathsf{g}_1 \bullet \mathsf{f}_1}(x), \quad \forall\, x \in \operatorname{dom}(\mathsf{g}_0 \bullet \mathsf{f}_0). \qquad \square$$

Lemma 3.2 **SetTP**$/\!\approx$ *is isomorphic to* **Set**. *In fact, the following diagram commutes*:

$$\begin{array}{ccc} \mathbf{SetTP} & \xrightarrow{S} & \mathbf{Set} \\ {\scriptstyle Q}\downarrow & \swarrow{\scriptstyle F} & \\ \mathbf{SetTP}/\!\approx & & \end{array} \tag{3.1}$$

where Q is the quotient functor and F is an isomorphism that maps sets to themselves and maps morphisms $f\colon X \to Y$ to $S^{-1}(\{f\})$.

Proof The object parts of S, P, and F map every set to itself, so the said object parts must commute. Every $\mathsf{f} \in \hom_{\mathbf{SetTP}}(X, Y)$ satisfies

$$Q(\mathsf{f}) = \{\mathsf{f}' \in \hom_{\mathbf{SetTP}}(X, Y) \mid \overleftarrow{\mathsf{f}'} = \overleftarrow{\mathsf{f}}\} = S^{-1}(\{\overleftarrow{\mathsf{f}}\}) = F(S(\mathsf{f})). \qquad \square$$

Similar to **Set**, $\emptyset$ is an initial object for **SetP**, **SetM**, and **SetTP** (see Example 2.5). Every singleton set is a terminal object for **SetTP** (see Example 2.3).

Proposition 3.3 *Given nonempty sets X, Y and any $\mathsf{f} \in \hom_{\mathbf{SetP}}(X, Y)$, f is a section if and only if f is total and $(f_0(x_0) = f_1(x_1) \implies x_0 = x_1)$ for all $x_0, x_1 \in X$ and $f_0, f_1 \in \bigcup_{x \in X} \langle \mathsf{f} \rangle(x)$.*

Proof ($\Longrightarrow$) Suppose that f is a section; there exists some polymap $\mathsf{g}\colon Y \Rrightarrow X$ such that $\mathsf{g} \bullet \mathsf{f} = 1_X^X$. By Lemma 2.7,

$$X = \operatorname{dom}(\mathsf{g} \bullet \mathsf{f}) \subseteq \operatorname{dom} \mathsf{f} \subseteq X,$$

and so f must be total. Consider any $x_0, x_1 \in X$ and $f_0, f_1 \in \bigcup_{x\in X}\langle\mathsf{f}\rangle(x)$ such that $f_0(x_0) = f_1(x_1)$. The condition that $\operatorname{dom}(\mathsf{g} \bullet \mathsf{f}) = \operatorname{dom}\mathsf{f}$ implies that $\langle\mathsf{g}\rangle(f_0(x_0)) \neq \emptyset$, so choose any $g \in \langle\mathsf{g}\rangle(f_0(x_0))$. It follows that $g\circ f_0 = \mathrm{id} = g\circ f_1$, and that

$$x_0 = \mathrm{id}(x_0) = g(f_0(x_0)) = g(f_1(x_1)) = \mathrm{id}(x_1) = x_1.$$

($\Longleftarrow$) Suppose that f is total and ($f_0(x_0) = f_1(x_1) \implies x_0 = x_1$) for any $x_0, x_1 \in X$ and $f_0, f_1 \in \bigcup_{x\in X}\langle\mathsf{f}\rangle(x)$. For any $f \in \bigcup_{x\in X}\langle\mathsf{f}\rangle(x)$, f must be injective, so let g_f denote the restriction of some retraction of f to $f(X)$. For any $y \in Y$, the family $\{g_f\}_{f\in\bigcup\langle\mathsf{f}\rangle(\overleftarrow{\mathsf{f}}^{-1}(\{y\}))}$ can be summed following [10] because for any $f_0, f_1 \in \bigcup\langle\mathsf{f}\rangle(\overleftarrow{\mathsf{f}}^{-1}(\{y\}))$ and $y' \in f_0(X)\cap f_1(X)$, observe that $f_0(g_{f_0}(y')) = y' = f_1(g_{f_1}(y'))$, thus $g_{f_0}(y') = g_{f_1}(y')$.

There exists some monomap $\mathsf{g}\colon Y \Rrightarrow X$, where $\operatorname{im}\mathsf{f} = \operatorname{dom}\mathsf{g}$ and $\langle\mathsf{g}\rangle(y) = \{g\}$ for some g such that

$$g|_R = \sum_{f\in\bigcup\langle\mathsf{f}\rangle(\overleftarrow{\mathsf{f}}^{-1}(\{y\}))} g_f, \tag{3.2}$$

where

$$R = \bigcup_{f\in\bigcup\langle\mathsf{f}\rangle(\overleftarrow{\mathsf{f}}^{-1}(\{y\}))} f(X).$$

To verify that $\mathsf{g}\bullet\mathsf{f} = 1_X^X$, observe that for any $x \in X$ and $f' \in \langle\mathsf{f}\rangle(x)$, $g_{f'}$ is a restriction of g, where $\langle\mathsf{g}\rangle(\overleftarrow{\mathsf{f}}(x)) = \{g\}$, thus $g\circ f' = \mathrm{id}$. Hence,

$$\langle\mathsf{g}\bullet\mathsf{f}\rangle(x) = \{\mathrm{id}\}. \qquad \square$$

Given a set A of maps from X to Y, let $\bigwedge A$ denote the map whose graph is the intersection of the graphs of the members of A. This is an extension of the intersection operator $\cdot$ from [10].

Proposition 3.4 *Given nonempty sets X, Y and any $\mathsf{f} \in \hom_{\mathbf{SetP}}(X, Y)$, f is a retraction if and only if there exists some $X' \subseteq \operatorname{dom}\mathsf{f}$ such that $\overleftarrow{\mathsf{f}}|_{X'}$ is surjective and $\bigwedge\langle\mathsf{f}\rangle(x)$ is surjective for all $x \in X'$.*

Proof ($\Longrightarrow$) Suppose that there exists some polymap $\mathsf{e}\colon Y \Rrightarrow X$ such that $\mathsf{f}\bullet\mathsf{e} = 1_Y^Y$; note that

$$\operatorname{dom}\mathsf{e} \subseteq Y = \operatorname{dom}(\mathsf{f}\bullet\mathsf{e}) = \overleftarrow{\mathsf{e}}^{-1}(\operatorname{dom}\mathsf{f}) \subseteq \operatorname{dom}\mathsf{e}.$$

Hence, $\operatorname{dom}\mathsf{e} = \overleftarrow{\mathsf{e}}^{-1}(\operatorname{dom}\mathsf{f})$, and thus, $\operatorname{im}\mathsf{e} \subseteq \operatorname{dom}\mathsf{f}$. By Lemma 2.7,

$$(\overleftarrow{\mathsf{f}}|_{\operatorname{im}\mathsf{e}})\circ\overleftarrow{\mathsf{e}} = \overleftarrow{\mathsf{f}\bullet\mathsf{e}} = \overleftarrow{1_Y^Y} = \mathrm{id},$$

thus, $\overleftarrow{\mathsf{f}}|_{\operatorname{im}\mathsf{e}}$ must be surjective. Consider any $x \in \operatorname{im}\mathsf{e} \subseteq \operatorname{dom}\mathsf{f}$. Choose any $f \in \langle\mathsf{f}\rangle(x)$, $y \in \overleftarrow{\mathsf{e}}^{-1}(\{x\})$, and $e \in \langle\mathsf{e}\rangle(y)$; $f \circ e = \mathrm{id} = f|_{\operatorname{im}\mathsf{e}} \circ e$ so e must be injective and $f|_{\operatorname{im}\mathsf{e}}$ must be surjective. For any $y' \in Y$ and $f' \in \langle\mathsf{f}\rangle(x)$, notice that

$$f'(e(y')) = y' = f(e(y')),$$

thus, $f|_{\operatorname{im}\mathsf{e}}$ is a restriction of f'. Therefore, $f|_{\operatorname{im}\mathsf{e}}$ is a restriction of $\bigwedge\langle\mathsf{f}\rangle(x)$, and both must be surjective.

($\Longleftarrow$) Suppose that there exists some $X' \subseteq \operatorname{dom}\mathsf{f}$ such that $\overleftarrow{\mathsf{f}}|_{X'}$ is surjective and $\bigwedge\langle\mathsf{f}\rangle(x)$ is surjective for all $x \in X'$. There exists some map $e\colon Y \to X'$ such that $(\overleftarrow{\mathsf{f}}|_{X'}) \circ e = \mathrm{id}$, because $(\overleftarrow{\mathsf{f}}|_{X'})$ is surjective. For any $x \in X'$, choose some map ε_x such that $(\bigwedge\langle\mathsf{f}\rangle(x)) \circ \varepsilon_x = \mathrm{id}$ (since $\bigwedge\langle\mathsf{f}\rangle(x)$ is surjective). For any $y \in Y$, define the map $\epsilon_y\colon Y \to X$ such that

$$\epsilon_y(y') = \begin{cases} e(y'), & y = y', \\ \varepsilon_{e(y)}(y'), & y \neq y'. \end{cases}$$

Given the monomap $\mathsf{e}\colon Y \Rrightarrow X$, where $\langle\mathsf{e}\rangle(y) = \{\epsilon_y\}$, it is verified that $\mathsf{f} \bullet \mathsf{e} = 1_Y^Y$. Observe that $\overleftarrow{\mathsf{e}}$ is simply a codomain expansion of e, so

$$\operatorname{dom}(\mathsf{f} \bullet \mathsf{e}) = \overleftarrow{\mathsf{e}}^{-1}(\operatorname{dom}\mathsf{f}) = e^{-1}(\operatorname{dom}\mathsf{f} \cap X') = Y.$$

Note that for all $y, y' \in Y$ and $f \in \langle\mathsf{f}\rangle(e(y))$,

$$\begin{aligned} f(\epsilon_y(y')) &= \begin{cases} f(e(y')), & y = y', \\ f(\varepsilon_{e(y)}(y')), & y \neq y', \end{cases} \\ &= \begin{cases} \overleftarrow{\mathsf{f}}|_{X'}(e(y')), & y = y', \\ (\bigwedge\langle\mathsf{f}\rangle(e(y)))(\varepsilon_{e(y)}(y')), & y \neq y', \end{cases} \\ &= \mathrm{id}(y') \\ &= y'. \end{aligned}$$

Hence, for any $y \in Y$,

$$\langle\mathsf{f} \bullet \mathsf{e}\rangle(y) = \langle\mathsf{f}\rangle(\overleftarrow{\mathsf{e}}(y)) \circledcirc \langle\mathsf{e}\rangle(y) = \{f \circ \epsilon_y \mid f \in \langle\mathsf{f}\rangle(e(y))\} = \{\mathrm{id}\}. \qquad \square$$

The inverse polymap is not, in general, a multiplicative inverse with respect to polymap composition. Some necessary conditions for polymaps to be sections or retractions are established as follows.

Theorem 3.5 *Given nonempty sets X, Y and some $\mathsf{f} \in \hom_{\mathbf{SetP}}(X, Y)$,*

(1) *if f is a section, then f is total and polyinjective, and $\overleftarrow{\mathsf{f}}$ is injective*;

(2) *if f is a retraction, then $\overleftarrow{\mathsf{f}}$ is surjective*;

(3) f *is an isomorphism if and only if* f *is total,* $\overleftarrow{\mathsf{f}}$ *is surjective, and any of the following equivalent statements hold*:

(a) f *is an invertible monomap*;

(b) f *is invertible and* $\mathsf{f}^{-1} \bullet \mathsf{f} = 1^X_{\mathrm{dom}\,\mathsf{f}}$;

(c) f *is invertible and* $\mathsf{f} \bullet \mathsf{f}^{-1} = 1^Y_{\mathrm{im}\,\mathsf{f}}$.

Proof (1) If f is a section, then by setting $f_0 = f_1$ in Proposition 3.3, it follows immediately that f is total and polyinjective. For any $x_0, x_1 \in \mathrm{dom}\,\mathsf{f}$ such that $\overleftarrow{\mathsf{f}}(x_0) = \overleftarrow{\mathsf{f}}(x_1)$, choose any $f_0' \in \langle \mathsf{f} \rangle (x_0)$ and $f_1' \in \langle \mathsf{f} \rangle (x_1)$. It can be seen that

$$f_0'(x_0) = \overleftarrow{\mathsf{f}}(x_0) = \overleftarrow{\mathsf{f}}(x_1) = f_1'(x_1),$$

and thus, $x_0 = x_1$. This implies that f is injective.

(2) If f is a retraction, then by Proposition 3.4, there exists some $X' \subseteq X$ such that $\overleftarrow{\mathsf{f}}|_{X'}$ is surjective. It follows that $\overleftarrow{\mathsf{f}}$ must also be surjective.

(3) (a), (b), and (c) are first shown to be equivalent.

(a) $\Longrightarrow$ (c) Suppose that f is a invertible monomap. For any $y \in \mathrm{im}\,\mathsf{f} = \mathrm{dom}\,(\mathsf{f} \bullet \mathsf{f}^{-1})$, let $\langle \mathsf{f}^{-1} \rangle (y) = \{g\}$. It follows that

$$\langle \mathsf{f} \bullet \mathsf{f}^{-1} \rangle (y) = \langle \mathsf{f} \rangle (\overleftarrow{\mathsf{f}^{-1}}(y)) \circledcirc \langle \mathsf{f}^{-1} \rangle (y) = \{g^{-1}\} \circledcirc \{g\} = \{\mathrm{id}\}.$$

Thus, $\mathsf{f} \bullet \mathsf{f}^{-1} = 1^Y_{\mathrm{im}\,\mathsf{f}}$.

(c) $\Longrightarrow$ (b) Suppose that $\mathsf{f} \bullet \mathsf{f}^{-1} = 1^Y_{\mathrm{im}\,\mathsf{f}}$. For any $x \in \mathrm{dom}\,\mathsf{f}$, $f \in \langle \mathsf{f} \rangle (x)$, and $g \in \langle \mathsf{f}^{-1} \rangle (\overleftarrow{\mathsf{f}}(x))$, note that by (c),

$$\begin{aligned} \{\mathrm{id}\} &= \langle \mathsf{f} \bullet \mathsf{f}^{-1} \rangle (\overleftarrow{\mathsf{f}}(x)) \\ &= \langle \mathsf{f} \rangle (\overleftarrow{\mathsf{f}^{-1}}(\overleftarrow{\mathsf{f}}(x))) \circledcirc \langle \mathsf{f}^{-1} \rangle (\overleftarrow{\mathsf{f}}(x)) \\ &= \langle \mathsf{f} \rangle (x) \circledcirc \langle \mathsf{f}^{-1} \rangle (\overleftarrow{\mathsf{f}}(x)) \\ &\ni f \circ g. \end{aligned}$$

Then $f \circ g = \mathrm{id}$. However, f and g are bijections because f and f^{-1} are invertible, so $g = f^{-1}$ and $g \circ f = \mathrm{id}$. Hence,

$$\langle \mathsf{f}^{-1} \bullet \mathsf{f} \rangle (x) = \langle \mathsf{f}^{-1} \rangle (\overleftarrow{\mathsf{f}}(x)) \circledcirc \langle \mathsf{f} \rangle (x) = \{\mathrm{id}\}.$$

(b) $\Longrightarrow$ (a) Suppose that $\mathsf{f}^{-1} \bullet \mathsf{f} = 1^X_{\mathrm{dom}\,\mathsf{f}}$; f is invertible. For any $x \in \mathrm{dom}\,\mathsf{f}$, choose any $g \in \langle \mathsf{f}^{-1} \rangle (\overleftarrow{\mathsf{f}}(x))$. Since f^{-1} must also be invertible, g is bijective. For any $f \in \langle \mathsf{f} \rangle (x)$,

$$\{\mathrm{id}\} = \langle \mathsf{f}^{-1} \bullet \mathsf{f} \rangle (x) = \langle \mathsf{f}^{-1} \rangle (\overleftarrow{\mathsf{f}}(x)) \circledcirc \langle \mathsf{f} \rangle (x) \ni g \circ f,$$

thus, $g \circ f = \mathrm{id}$ and $f = g^{-1}$. Therefore, every member of $\langle \mathsf{f} \rangle(x)$ is equivalent to g^{-1}, and f must be a monomap.

Next, it is proven that f is an isomorphism if and only if f is a total invertible monomap such that $\overleftarrow{\mathsf{f}}$ is surjective.

($\Longleftarrow$) If f is a total invertible monomap, then

$$\mathsf{f}^{-1} \bullet \mathsf{f} = 1^X_{\mathrm{dom}\,\mathsf{f}} = 1^X_X.$$

($\Longrightarrow$) If f is an isomorphism, then it is both a section and a retraction. By (1) and (2), f is total and polyinjective, and $\overleftarrow{\mathsf{f}}$ is bijective. By Proposition 3.4, there exists some $X' \subseteq \mathrm{dom}\,\mathsf{f}$ such that $\overleftarrow{\mathsf{f}}|_{X'}$ is surjective and $\bigwedge \langle \mathsf{f} \rangle(x)$ is surjective for all $x \in X'$. However, since $\overleftarrow{\mathsf{f}}$ is bijective, X' must be equivalent to X or else $\overleftarrow{\mathsf{f}}|_{X'}$ would not be surjective. Hence, $\overleftarrow{\mathsf{f}}$ is surjective. Clearly, f must be injective for any $x \in X = \mathrm{dom}\,\mathsf{f}$ and $f \in \langle \mathsf{f} \rangle(x)$; moreover, $\bigwedge \langle \mathsf{f} \rangle(x)$ is a restriction of f so f must also be surjective. Therefore, f is bijective. This leads to the invertibility of f. Choose any two bijections $f_0, f_1 \in \langle \mathsf{f} \rangle(x)$; by Proposition 3.3, $f_0^{-1}(y) = f_1^{-1}(y)$ for all $y \in Y$; this means that $f_0^{-1} = f_1^{-1}$, and consequently, $f_0 = f_1$, and $|\langle \mathsf{f} \rangle(x)| = 1$. This concludes that f must be a monomap. □

4 Algebraic structures from polymap sum

Algebraic structures involving polymaps are constructed by considering a sum over arbitrary families of polymaps. This operation is in turn based on the restriction of the relation union + from [10] to functional binary relations.

A family $\{f_i \colon X_i \to Y_i\}_{i \in I}$ of maps is called *summable* if the union of their graphs is a functional binary relation $R \subseteq (\bigcup_i X_i) \times (\bigcup_i Y_i)$, in which the sum of the family is denoted as the map $\sum_i f_i \colon \bigcup_i X_i \to \bigcup_i Y_i$ whose graph is R.

4.1 Polymap sum

Definition 4.1 Given a family $\{\mathsf{f}_i \colon X \Rrightarrow Y\}_{i \in I}$ of polymaps, the family is called *summable* if $\{\overleftarrow{\mathsf{f}_i}\}_{i \in I}$ is a summable family of maps. If $\{\mathsf{f}_i\}_{i \in I}$ is summable, then its *sum* is defined as

$$\sum_i \mathsf{f}_i = \Big(X, x \mapsto \bigcup_i \langle \mathsf{f}_i \rangle(x), Y\Big),$$

which can be verified to be a polymap. In the case where $I = \{0, 1\}$, the sum can also be written as $\mathsf{f}_0 + \mathsf{f}_1$.

When well-defined, polymap summation is clearly idempotent, commutative, and associative, with each $\mathsf{0}^Y_X = \mathsf{C}_\emptyset \colon X \Rrightarrow Y$ as an additive identity. Some immediate consequences are as follows.

Lemma 4.2 *If* $\{\mathsf{f}_i\}_{i\in I}$ *is summable, then*

$$\mathrm{dom}\Big(\sum_i \mathsf{f}_i\Big) = \bigcup_i \mathrm{dom}\,\mathsf{f}_i, \quad \overleftarrow{\sum_i \mathsf{f}_i} = \sum_i \overleftarrow{\mathsf{f}_i}.$$

4.2 Groups and related structures

Consider a set X. Similar to the transformation semigroups and monoid arising from subsets of $\hom_{\mathbf{Set}}(X,X)$, semigroups and monoids can arise from subsets of $\hom_{\mathbf{SetP}}(X,X)$, including the endomorphism semigroup $\hom_{\mathbf{SetP}}(X,X)$ itself. By Cayley's Theorem [14] on semigroups, every semigroup of polymaps must be isomorphic to a semigroup of maps.

Analogies to the symmetric group $\mathrm{Sym}(X)$ on X are of greater interest, because they inherit much of its well-behaved structure. The set of all total monomaps $\mathsf{f}\colon X \Rrightarrow X$ such that

$$\{\overleftarrow{\mathsf{f}}\} \cup \bigcup_{x\in X} \langle \mathsf{f}\rangle(x) \subseteq \mathrm{Sym}(X)$$

forms a group under polymap composition and inverse. By Theorem 3.5, this group is

$$\mathrm{Aut}_{\mathbf{SetP}}(X) = \mathrm{Aut}_{\mathbf{SetTP}}(X) = \mathrm{Aut}_{\mathbf{SetM}}(X),$$

and it is a subgroup of the group of invertible monomaps (from X to X) under polymap composition and inverse.

Let $\mathsf{f}\colon X \Rrightarrow Y$, $\mathsf{g}\colon Y \Rrightarrow Z$, and $\mathsf{h}\colon Z \Rrightarrow W$ be polymaps. Polymap composition distributes over polymap sum.

Theorem 4.3 *If* $\{\mathsf{g}_i\colon Y \Rrightarrow Z\}_{i\in I}$ *is summable, then* $\{\mathsf{g}_i \bullet \mathsf{f}\}_{i\in I}$ *and* $\{\mathsf{h}\bullet \mathsf{g}_i\}_{i\in I}$ *are summable, and*

$$\Big(\sum_i \mathsf{g}_i\Big)\bullet \mathsf{f} = \sum_i (\mathsf{g}_i \bullet \mathsf{f}), \quad \mathsf{h}\bullet\Big(\sum_i \mathsf{g}_i\Big) = \sum_i (\mathsf{h}\bullet \mathsf{g}_i).$$

Proof Consider any $i_0, i_1 \in I$. For any $x \in \mathrm{dom}\,(\mathsf{g}_{i_0}\bullet \mathsf{f}) \cap \mathrm{dom}\,(\mathsf{g}_{i_1}\bullet \mathsf{f})$, Lemma 2.7 requires that $\overleftarrow{\mathsf{g}_{i_0}\bullet \mathsf{f}}(x) = \overleftarrow{\mathsf{g}_{i_0}}(\overleftarrow{\mathsf{f}}(x))$. Since $\{\mathsf{g}_i\colon Y \Rrightarrow Z\}_{i\in I}$ is summable,

$$\overleftarrow{\mathsf{g}_{i_0}}(\overleftarrow{\mathsf{f}}(x)) = \overleftarrow{\mathsf{g}_{i_1}}(\overleftarrow{\mathsf{f}}(x)) = \overleftarrow{\mathsf{g}_{i_1}\bullet \mathsf{f}}(x).$$

Hence, $\{\mathsf{g}_i \bullet \mathsf{f}\}_{i\in I}$ is summable. Similarly, for any $y \in \mathrm{dom}\,(\mathsf{h}\bullet \mathsf{g}_{i_0}) \cap \mathrm{dom}\,(\mathsf{h}\bullet \mathsf{g}_{i_1})$, Lemma 2.7 requires that $\overleftarrow{\mathsf{h}\bullet \mathsf{g}_{i_0}}(y) = \overleftarrow{\mathsf{h}}(\overleftarrow{\mathsf{g}_{i_0}}(y))$. Since $\{\mathsf{g}_i\colon Y \Rrightarrow Z\}_{i\in I}$ is summable,

$$\overleftarrow{\mathsf{h}}(\overleftarrow{\mathsf{g}_{i_0}}(y)) = \overleftarrow{\mathsf{h}}(\overleftarrow{\mathsf{g}_{i_1}}(y)) = \overleftarrow{\mathsf{h}\bullet \mathsf{g}_{i_1}}(x).$$

Hence, $\{\mathsf{h} \bullet \mathsf{g}_i\}_{i\in I}$ is summable. By Lemma 4.2,

$$\begin{aligned}
\operatorname{dom}\Big(\Big(\sum_i \mathsf{g}_i\Big) \bullet \mathsf{f}\Big) &= \overleftarrow{\mathsf{f}}^{-1}\Big(\bigcup_i \operatorname{dom} \mathsf{g}_i\Big) \\
&= \bigcup_i \Big(\overleftarrow{\mathsf{f}}^{-1}(\operatorname{dom} \mathsf{g}_i)\Big) \\
&= \bigcup_i \operatorname{dom}(\mathsf{g}_i \bullet \mathsf{f}) \\
&= \operatorname{dom}\Big(\sum_i (\mathsf{g}_i \bullet \mathsf{f})\Big).
\end{aligned}$$

For any $x \in \bigcup_i \overleftarrow{\mathsf{f}}^{-1}(\operatorname{dom} \mathsf{g}_i) \subseteq \operatorname{dom} \mathsf{f}$,

$$\begin{aligned}
\Big\langle\Big(\sum_i \mathsf{g}_i\Big) \bullet \mathsf{f}\Big\rangle(x) &= \Big\langle\sum_i \mathsf{g}_i\Big\rangle(\overleftarrow{\mathsf{f}}(x)) \circledcirc \langle \mathsf{f}\rangle(x) \\
&= \Big(\bigcup_i \langle \mathsf{g}_i\rangle(\overleftarrow{\mathsf{f}}(x))\Big) \circledcirc \langle \mathsf{f}\rangle(x) \\
&= \bigcup_i \Big(\langle \mathsf{g}_i\rangle(\overleftarrow{\mathsf{f}}(x)) \circledcirc \langle \mathsf{f}\rangle(x)\Big) \\
&= \bigcup_i \langle \mathsf{g}_i \bullet \mathsf{f}\rangle(x) \\
&= \Big\langle\sum_i (\mathsf{g}_i \bullet \mathsf{f})\Big\rangle(x).
\end{aligned} \tag{4.1}$$

Similarly, by Lemma 4.2,

$$\begin{aligned}
\operatorname{dom}\Big(\mathsf{h} \bullet \Big(\sum_i \mathsf{g}_i\Big)\Big) &= \overleftarrow{\sum_i \mathsf{g}_i}^{-1}(\operatorname{dom} \mathsf{h}) \\
&= \bigcup_i \overleftarrow{\mathsf{g}_i}^{-1}(\operatorname{dom} \mathsf{h}) \\
&= \bigcup_i \operatorname{dom}(\mathsf{h} \bullet \mathsf{g}_i) \\
&= \operatorname{dom}\Big(\sum_i (\mathsf{h} \bullet \mathsf{g}_i)\Big).
\end{aligned}$$

For any

$$y \in \bigcup_{i\in I} \overleftarrow{\mathsf{g}_i}^{-1}(\operatorname{dom} \mathsf{h}) \subseteq \operatorname{dom}\Big(\sum_{i\in I} \mathsf{g}_i\Big),$$

let

$$J = \{j \in I \mid y \in \operatorname{dom} \mathsf{g}_j\}.$$

It follows that

$$\begin{aligned}\Big\langle \mathsf{h} \bullet \Big(\sum_{i\in I} \mathsf{g}_i\Big)\Big\rangle(y) &= \langle \mathsf{h}\rangle\Big(\overleftarrow{\sum_{i\in I} \mathsf{g}_i}(y)\Big) \circledcirc \Big(\bigcup_{j\in J} \langle \mathsf{g}_j\rangle(y)\Big)\\ &= \bigcup_{j\in J}(\langle \mathsf{h}\rangle(\overleftarrow{\mathsf{g}_j}(y)) \circledcirc \langle \mathsf{g}_j\rangle(y))\\ &= \bigcup_{j\in J}\langle \mathsf{h}\bullet \mathsf{g}_j\rangle(y)\\ &= \Big\langle \sum_{i\in I}(\mathsf{h}\bullet \mathsf{g}_i)\Big\rangle(y). \qquad \square\end{aligned}$$

While Theorem 3.5 shows that in general the inverse polymap is not a multiplicative inverse with respect to polymap composition, polymap inversion does satisfy many properties of involutions compatible with operations on ring-like structures.

Lemma 4.4 *For any sets $X' \subseteq X$ and Y, it holds that*

$$(1^X_{X'})^{-1} = 1^X_{X'}, \quad (0^Y_X)^{-1} = 0^X_Y.$$

Polymap inversion also has properties analogous to antihomomorphism.

Proposition 4.5 *If f and g are invertible, then $\mathsf{g}\bullet\mathsf{f}$ is invertible and $(\mathsf{g}\bullet\mathsf{f})^{-1} = \mathsf{f}^{-1}\bullet \mathsf{g}^{-1}$.*

Proof Since every element of $\bigcup_{x\in X}\langle \mathsf{g}\bullet\mathsf{f}\rangle(x)$ is the composite of two bijections, $\mathsf{g}\bullet\mathsf{f}$ is polybijective. By Lemma 2.7,

$$\overleftarrow{\mathsf{g}\bullet\mathsf{f}} = \overleftarrow{\mathsf{g}} \circ (\overleftarrow{\mathsf{f}}|_{\overleftarrow{\mathsf{f}}^{-1}(\mathrm{dom}\,\mathsf{g})})$$

must be injective. Hence, $\mathsf{g}\bullet\mathsf{f}$ is invertible.

Note that $\overleftarrow{\mathsf{g}}$ and $\overleftarrow{\mathsf{g}^{-1}}$ are injective. By Lemma 2.11, the graph of $\overleftarrow{\mathsf{g}^{-1}}$ is the relation inverse of the graph of $\overleftarrow{\mathsf{g}}$ so for all $Y' \subseteq Y$ and $Z' \subseteq Z$ such that $\overleftarrow{\mathsf{g}}(Y') = Z'$, we have $\overleftarrow{\mathsf{g}^{-1}}(Z') = Y'$, which means that $\overleftarrow{\mathsf{g}^{-1}}^{-1}(Y') = Z' = \overleftarrow{\mathsf{g}}(Y')$.

By Lemmas 2.10 and 2.7,

$$\begin{aligned}\mathrm{dom}\,((\mathsf{g}\bullet\mathsf{f})^{-1}) &= \mathrm{im}\,(\mathsf{g}\bullet\mathsf{f}) &(4.2)\\ &= \overleftarrow{\mathsf{g}\bullet\mathsf{f}}(\mathrm{dom}\,(\mathsf{g}\bullet\mathsf{f})) &(4.3)\\ &= \overleftarrow{\mathsf{g}}(\overleftarrow{\mathsf{f}}(\overleftarrow{\mathsf{f}}^{-1}(\mathrm{dom}\,\mathsf{g}))) &(4.4)\\ &= \overleftarrow{\mathsf{g}}(\mathrm{im}\,\mathsf{f}\cap \mathrm{dom}\,\mathsf{g}) &(4.5)\\ &= \overleftarrow{\mathsf{g}^{-1}}^{-1}(\mathrm{im}\,\mathsf{f}\cap \mathrm{dom}\,\mathsf{g}) &(4.6)\\ &= \overleftarrow{\mathsf{g}^{-1}}^{-1}(\mathrm{dom}\,(\mathsf{f}^{-1})) &(4.7)\\ &= \mathrm{dom}\,(\mathsf{f}^{-1}\bullet\mathsf{g}^{-1}). &(4.8)\end{aligned}$$

For any $z \in \operatorname{im}(\mathsf{g} \bullet \mathsf{f})$, there exists exactly one $x \in X$ such that $\overleftarrow{\mathsf{g} \bullet \mathsf{f}}(x) = z$, due to the injectivity of $\overleftarrow{\mathsf{g} \bullet \mathsf{f}}$. Similarly, $\overleftarrow{\mathsf{f}}$ is injective so $\overleftarrow{\mathsf{f}}^{-1}(\{\overleftarrow{\mathsf{f}}(x)\}) = \{x\}$. Note that $\overleftarrow{\mathsf{g}}(\overleftarrow{\mathsf{f}}(x)) = \overleftarrow{\mathsf{g} \bullet \mathsf{f}}(x) = z$ so $\overleftarrow{\mathsf{g}^{-1}}(z) = \overleftarrow{\mathsf{f}}(x)$.

$$\begin{aligned}
\langle (\mathsf{g} \bullet \mathsf{f})^{-1} \rangle(z) &= \{e^{-1} \mid e \in \langle \mathsf{g} \bullet \mathsf{f} \rangle(x)\} \\
&= \{f^{-1} \circ g^{-1} \mid f \in \langle \mathsf{f} \rangle(x),\, g \in \langle \mathsf{g} \rangle(\overleftarrow{\mathsf{f}}(x))\} \\
&= \{f^{-1} \mid f \in \langle \mathsf{f} \rangle(x)\} \circledcirc \{g^{-1} \mid g \in \langle \mathsf{g} \rangle(\overleftarrow{\mathsf{f}}(x))\} \\
&= \langle \mathsf{f}^{-1} \rangle(\overleftarrow{\mathsf{f}}(x)) \circledcirc \langle \mathsf{g}^{-1} \rangle(z) \\
&= \langle \mathsf{f}^{-1} \rangle(\overleftarrow{\mathsf{g}^{-1}}(z)) \circledcirc \langle \mathsf{g}^{-1} \rangle(z) \\
&= \langle \mathsf{f}^{-1} \bullet \mathsf{g}^{-1} \rangle(z).
\end{aligned}$$

□

Polymap inversion also distributes over the polymap sum.

Proposition 4.6 *Given a summable family of polymaps* $\{\mathsf{f}_i\}_{i \in I}$ *such that* $\sum_i \mathsf{f}_i$ *is invertible, each* f_i *is also invertible, the family* $\{\mathsf{f}_i^{-1}\}_{i \in I}$ *is summable, and* $(\sum_i \mathsf{f}_i)^{-1} = \sum_i \mathsf{f}_i^{-1}$.

Proof Since $\sum_i \mathsf{f}_i$ is invertible, $\overleftarrow{\sum_i \mathsf{f}_i}$ must be injective. It also follows that every member of

$$\bigcup_{x \in X} \Big\langle \sum_i \mathsf{f}_i \Big\rangle(x) = \bigcup_i \bigcup_{x \in X} \langle \mathsf{f}_i \rangle(x)$$

must be bijective. Naturally, for each $i \in I$, $\overleftarrow{\mathsf{f}_i} = \overleftarrow{\sum_i \mathsf{f}_i}|_{\operatorname{dom} \mathsf{f}_i}$ is injective and every member of $\bigcup_{x \in X} \langle \mathsf{f}_i \rangle(x)$ is bijective, and thus, f_i is invertible.

For any $i_0, i_1 \in I$ and

$$y \in \operatorname{dom} \mathsf{f}_{i_0}^{-1} \cap \operatorname{dom} \mathsf{f}_{i_1}^{-1} = \operatorname{im} \mathsf{f}_{i_0} \cap \operatorname{im} \mathsf{f}_{i_1}$$

(Lemma 2.10), there exists exactly one x such that

$$\overleftarrow{\sum_i} \mathsf{f}_i(x) = \overleftarrow{\mathsf{f}_{i_0}}(x) = \overleftarrow{\mathsf{f}_{i_1}}(x) = y.$$

Choose any $f_0 \in \langle \mathsf{f}_{i_0} \rangle(x)$ and $f_1 \in \langle \mathsf{f}_{i_1} \rangle(x)$; note that both are bijections, and $f_0(x) = y = f_1(x)$. This also implies that

$$f_0^{-1} \in \langle \mathsf{f}_{i_0}^{-1} \rangle(y), \quad f_1^{-1} \in \langle \mathsf{f}_{i_1}^{-1} \rangle(y),$$

and thus,

$$\overleftarrow{\mathsf{f}_{i_0}^{-1}}(y) = f_0^{-1}(y) = x = f_1^{-1}(y) = \overleftarrow{\mathsf{f}_{i_1}^{-1}}(y).$$

Hence, $\{\mathsf{f}_i^{-1}\}_{i \in I}$ is summable. By Lemma 2.10,

$$\operatorname{dom}\Big(\Big(\sum_i \mathsf{f}_i\Big)^{-1}\Big) = \operatorname{im}\Big(\sum_i \mathsf{f}_i\Big) = \bigcup_i \operatorname{im} \mathsf{f}_i = \bigcup_i \operatorname{dom} \mathsf{f}_i^{-1} = \operatorname{dom}\Big(\sum_i \mathsf{f}_i^{-1}\Big).$$

Finally, for any $y' \in \bigcup_i \operatorname{im} \mathsf{f}_i$, there exists a unique x' such that $\overleftarrow{\sum_i \mathsf{f}_i}(x') = y'$.

$$\begin{aligned}
\Big\langle \Big(\sum_i \mathsf{f}_i \Big)^{-1} \Big\rangle (y') &= \{(f')^{-1} \mid f' \in \langle \sum_i \mathsf{f}_i \rangle (x')\} \\
&= \bigcup_i \{f^{-1} \mid f \in \langle \mathsf{f}_i \rangle (x')\} \\
&= \bigcup_i \langle \mathsf{f}_i^{-1} \rangle (y') \\
&= \Big\langle \sum_i \mathsf{f}_i^{-1} \Big\rangle (y').
\end{aligned}$$

□

Given a complete $\sum$-semilattice $F \subseteq \hom_{\mathbf{SetP}}(X, Y)$ of which every member is invertible, Proposition 4.6 guarantees that the involution $\mathsf{f} \mapsto \mathsf{f}^{-1}$ is a complete semilattice isomorphism from F to $\{\mathsf{f}^{-1} \mid \mathsf{f} \in F\}$.

As polymaps contain more information than maps, there are algebraic structures that are peculiar to polymaps and are not direct analogues of algebraic structures on maps. Given a set X, Theorem 4.3 guarantees the existence of the following structure, similar to that described in [6], especially [9], where the involution is the relation inverse.

Lemma 4.7 *The set*

$$A = \{\mathsf{f} \in \hom_{\mathbf{SetP}}(X, X) \mid \mathsf{f} \textit{ is polybijective}, \overleftarrow{\mathsf{f}} \textit{ is an inclusion map}\}$$

forms an idempotent involution semiring $\mathbf{ID}(X)$ *with* $+$ *as addition,* $\bullet$ *as multiplication, polymap inverse as the involution, as well as* $\mathsf{0}_X^X$ *and* $\mathsf{1}_X^X$ *as the additive and multiplicative identities, respectively.*

Proof Noting that $+$ and $\bullet$ preserve polybijectivity, it can be seen that $(A, +)$ is a commutative idempotent semigroup (equivalently, a $+$-semilattice). By Theorem 4.3, $\bullet$ distributes over $+$, and $(A, \bullet)$ is a semigroup. Moreover, if $\mathsf{f}, \mathsf{g} \in A$, then Proposition 4.6 gives

$$(\mathsf{f} + \mathsf{g})^{-1} = \mathsf{f}^{-1} + \mathsf{g}^{-1}$$

and Proposition 4.5 gives

$$(\mathsf{g} \bullet \mathsf{f})^{-1} = \mathsf{f}^{-1} \bullet \mathsf{g}^{-1}.$$

Lemma 4.4 also implies that

$$(\mathsf{0}_X^X)^{-1} = \mathsf{0}_X^X, \quad (\mathsf{1}_X^X)^{-1} = \mathsf{1}_X^X.$$

□

The poset $(\mathbf{ID}(X), \leqslant)$ is a complete $\sum$-semilattice with least element $\mathsf{0}_X^X$ and greatest element i such that $\langle \mathsf{i} \rangle (x) = \operatorname{Aut}_{\mathbf{Set}^*}(X, x)$, the automorphism group of (X, x) in $\mathbf{Set}^*$.

Remark 4.8 Many of the algebraic structures of polymaps constructed above are natural extensions of their map analogues, which are their "projections" via the operator $\mathsf{f} \mapsto \overleftarrow{\mathsf{f}}$. This suggests a generalization of polymaps to "higher dimensions"; an n-polymap f from X to Y could be inductively defined as its projection, the map $\overleftarrow{\mathsf{f}} : X' \to Y$, with each $x \in X'$ assigned a set F of $(n-1)$-polymaps from X to Y such that $\overleftarrow{\mathsf{f}}(x) = \overleftarrow{\mathsf{f}'}(x)$ for all $\mathsf{f}' \in F$. A 0-polymap would be a map, and a 1-polymap would be a polymap.

This might allow the construction of analogues of familiar algebraic structures for general n-polymaps.

4.3 Partial order and semilattices

Given sets X and Y, the partial order $\leqslant$ restricted to $\hom_{\mathbf{SetP}}(X, Y)$ is that which is induced by the idempotent addition. That is,

$$\mathsf{f}_0 \leqslant \mathsf{f}_1 \iff \mathsf{f}_0 + \mathsf{f}_1 = \mathsf{f}_1.$$

Lemma 4.9 *Given some* $\mathsf{f} \in \hom_{\mathbf{SetP}}(X, Y)$, *define*

$$B = \{\mathsf{f}' \in \hom_{\mathbf{SetP}}(X, Y) \mid \mathsf{f}' \leqslant \mathsf{f}\}.$$

Then the poset $(B, \leqslant)$ *is a complete* $\sum$*-semilattice, called the* subpolymap semilattice *of* f.

Proof Consider any $F \subseteq B$, $\mathsf{f}_0, \mathsf{f}_1 \in F$, and $x \in \operatorname{dom} \mathsf{f}_0 \cap \operatorname{dom} \mathsf{f}_1$. Clearly, $\mathsf{f}_0, \mathsf{f}_1 \leqslant \mathsf{f}$, and so

$$\overleftarrow{\mathsf{f}_0}(x) = \overleftarrow{\mathsf{f}}(x) = \overleftarrow{\mathsf{f}_1}(x).$$

Hence, F is summable, and $\sum F \in B$. Therefore, F is a complete $\sum$-semilattice. □

Given a complete semilattice $G \subseteq \hom_{\mathbf{SetP}}(Y, Z)$, Theorem 4.3 guarantees that the "polymap actions" (or "composition operators") $\mathsf{g} \mapsto \mathsf{g} \bullet \mathsf{f}$ and $\mathsf{g} \mapsto \mathsf{h} \bullet \mathsf{g}$ are complete semilattice homomorphisms from G to the subpolymap semilattices of $(\sum G) \bullet \mathsf{f}$ and $\mathsf{h} \bullet (\sum G)$, respectively.

5 Polymaps as categories

A polymap $\mathsf{f} \colon X \Rrightarrow Y$ may be regarded as a category by regarding $\langle \mathsf{f} \rangle(x)$ as a hom-set for each $x \in X$. In fact, since every member of $\langle \mathsf{f} \rangle(x)$ maps x to $\overleftarrow{\mathsf{f}}(x)$,

$$\langle \mathsf{f} \rangle(x) \subseteq \hom_{\mathbf{Set}^*}((X, x), (Y, \overleftarrow{\mathsf{f}}(x))).$$

This observation motivates the interpretation of polymaps as small subcategories of $\mathbf{Set}^*$ as presented below.

Consider a map C from the class of all polymaps to the class of small subcategories of $\mathbf{Set}^*$, such that C maps every polymap $\mathsf{f} \colon X \to Y$ to a small subcategory $\mathbf{C}$ (called the *categorical form* of f) of $\mathbf{Set}^*$, where the objects of $\mathbf{C}$ are

$$\mathrm{Ob}(\mathbf{C}) = \{(X \times \{0\}, (x, 0)) \mid x \in X\} \cup \{(Y \times \{1\}, (y, 1)) \mid y \in Y\}$$

and if an (injective) operation $\widehat{\ }$ is defined on maps such that any map $f\colon X \to Y$ corresponds to $\widehat{f}\colon X \times \{0\} \to Y \times \{1\}$, where $\widehat{f}(x, 0) = (f(x), 1)$, then

$$\hom_{\mathbf{C}}((X_0, (x, 0)), (Y_1, (y, 1))) = \begin{cases} \{\widehat{f} \mid f \in \langle \mathsf{f} \rangle(x)\}, & \overleftarrow{\mathsf{f}}(x) = y, \\ \emptyset, & \overleftarrow{\mathsf{f}}(x) \neq y, \end{cases} \tag{5.1}$$

$$\hom_{\mathbf{C}}((Y_1, (y, 1)), (X_0, (x, 0))) = \emptyset, \tag{5.2}$$

$$\hom_{\mathbf{C}}((Z, (z, i)), (Z, (z', i))) = \begin{cases} \{\mathrm{id}\}, & z = z', \\ \emptyset, & z \neq z'. \end{cases} \tag{5.3}$$

Lemma 5.1 *$C(\mathsf{f})$ is a category for any polymap f.*

Proof By (5.3), each object $A \in \mathrm{Ob}(C(\mathsf{f}))$ has an identity morphism because $\{\mathrm{id}\} = \hom_{C(\mathsf{f})}(A, A)$. Consider the morphisms

$$(X', (x, i)) \xrightarrow{f'} (Y', (y, j)) \xrightarrow{g'} (Z', (z, k)) \xrightarrow{h'} (W', (w, l)).$$

By (5.2), $0 \leqslant i \leqslant j \leqslant k \leqslant l \leqslant 1$. Hence, at most one $e \in \{f', g', h'\}$ is not the identity map, in which case

$$h' \circ (g' \circ f') = e = (h' \circ g') \circ f'. \qquad \square$$

Note that monomaps are precisely the polymaps with thin categorical forms. C is clearly injective, allowing the representation of polymaps as small subcategories of $\mathbf{Set}^*$, which are called *categorical polymaps.* Lemma 5.2 below justifies the terms "subpolymap", "wide subpolymap", and "full subpolymap". Recall that a subcategory is *wide* when it inherits all of the objects from its parent space, and *full* when it inherits all morphisms between every pair of inherited objects [17].

Lemma 5.2 *Given polymaps $\mathsf{f}_0, \mathsf{f}_1\colon X \rightrightarrows Y$, f_0 is a subpolymap (resp. wide subpolymap, full subpolymap) of f_1 if and only if $C(\mathsf{f}_0)$ is a subcategory (resp. wide subcategory, full subcategory) of $C(\mathsf{f}_1)$.*

The inverse polymap relates to the opposite category.

Lemma 5.3 *If f is invertible, then $C(\mathsf{f}^{-1})$ is isomorphic to $C(\mathsf{f})^{\mathrm{op}}$.*

Proof Construct the required isomorphism $F\colon C(\mathsf{f})^{\mathrm{op}} \to C(\mathsf{f}^{-1})$ such that it maps each object $(X \times \{i\}, (x, i))$ to $(X \times \{1-i\}, (x, 1-i))$. When F is restricted to each hom-set $\hom_{C(\mathsf{f})^{\mathrm{op}}}((Y', (y, j)), (X', (x, i)))$, set F as a constant map with value id if $i = j$. Otherwise, if $i < j$ and $\overleftarrow{\mathsf{f}}(x) = y$, then F maps each f' to $\widehat{f^{-1}}$ where $\widehat{f} = f'$. $\square$

References

1. Belcastro S, Hull T C. Modelling the folding of paper into three dimensions using affine transformations. Linear Algebra Appl, 2002, 348: 273–282
2. Börger R. Connectivity spaces and component categories. In: Categorical Topology, Proc Int Conf, Toledo/Ohio 1983. Sigma Ser Pure Math, 5. 1984, 71–89
3. Cheng H Y, Cheong K H. Designing crease patterns for polyhedra by composing right frusta. Comput -Aided Design, 2012, 44: 331–342
4. Crvenković S, Dolinka I. Varieties of involution semigroups and involution semirings: a survey. Bull Soc Math Banja Luka, 2002, 9: 7–47
5. De Morgan A. On the syllogism: IV, and the logic of relations. T Cambridge Philos Soc, 1864, 10: 331–358
6. Dolinka I. Idempotent distributive semirings with involution. Internat J Algebra Comput, 2003, 13: 597–625
7. Dugowson S. On connectivity spaces. Cah Topol Géom Différ Catég, 2010, 51: 282–315
8. Easdown D, Munn W D. On semigroups with involution. Bull Aust Math Soc, 1993, 48: 93–100
9. Ésik Z, Bernátsky L. Equational properties of Kleene algebras of relations with conversion. Theoret Comput Sci, 1995, 137: 237–251
10. Givant S. The calculus of relations as a foundation for mathematics. J Automat Reason, 2006, 37: 277–322
11. Mac Lane S. Categories for the Working Mathematician. Berlin: Springer, 1978
12. Maddux R D. The origin of relation algebras in the development and axiomatization of the calculus of relations. Studia Logica, 1991, 50: 421–455
13. Muscat J, Buhagiar D. Connective spaces. Mem Fac Sci Eng, Shimane Univ Ser B Math Sci, 2006, 39: 1–13
14. Nummela E C. Cayley's theorem for topological groups. Amer Math Monthly, 1980, 87: 202–203
15. Reyes M L. Obstructing extensions of the functor spec to noncommutative rings. Israel J Math, 2012, 192: 667–698
16. Tarski A. On the calculus of relations. J Symbolic Logic, 1941, 6: 73–89
17. Taylor P. Practical Foundations of Mathematics. Cambridge: Cambridge University Press, 1999