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Occurrence of complementary processes in parrondo's paradox

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HIGHLIGHTS

- Condition for a classical Parrondo process to have a complementary process is proven.
- Numerical simulation of the parameter space was done based on the condition.
- More than half of losing Parrondo processes are found to have a complementary process.

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ABSTRACT

Parrondo's paradox involves two losing processes producing a winning outcome. We analyze the paradox with an original and novel method in which we start with one process and seek to construct a complementary process to achieve the paradox. We then derive a general condition for the classical Parrondo game to have a complementary process. Numerical simulation predicts that approximately two-thirds of such losing games satisfy the required condition. This suggests the common occurrence of the paradox, indicative of many potentially undiscovered applications in real-life scenarios involving stochastic processes.

1. Introduction

Parrondo's paradox [1,2] involves two losing games being played in a random or periodic alternation to produce a winning outcome. Chaotic switching has also been found to achieve the paradox [3]. Parrondo's paradox was discovered from investigations in the Brownian ratchet [4–8], where a Brownian particle shows systematic movement through switching between two different potentials.

There have been investigations on Parrondo games, such as determining optimal strategies and sequences [9–13]. The paradox has found numerous applications in quantum games and models [14–21]. The paradox is also found in chaotic systems [22–25], where a phenomenon of 'order from disorder' is apparent. Another major field of application of the paradox arises from population modeling and evolutionary biology [26–28]. Research in the field has mainly focused on having two processes interact to produce the counterintuitive result. Instead of considering two processes as a whole, we begin with one process, and seek to construct a complementary process for it.

In order to achieve Parrondo's paradox in a population model [27], it may not be easy to adjust or alter the process of stochastically switching phenotypes, and looking for two processes to achieve the paradox is not an easy task. However,

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we may introduce an artificial process through human intervention to achieve the paradox with another in nature. Human intervention could come in the form of artificially controlled lighting, temperature or even pH levels. This method is more plausible, and illustrates our approach of constructing a complementary process instead of simultaneously considering two processes. We have previously analyzed the paradox by imposing a condition that two out of three of the processes' parameters sum up to a constant value [29].

In this paper, we have provided a general condition for every losing Parrondo process in order for it to have a complementary process. We will first review the Parrondo games in our analysis, and understand its winning conditions. We then proceed to our theoretical result in the next section, along with a proof. We find that the conditions can be aptly summarized in a single inequality. Finally, we investigate the occurrence of the paradox by carrying out numerical simulation on the parameter space of the Parrondo games. It was found that approximately two-thirds of losing Parrondo games can have a complementary process, demonstrating the possibility of achieving Parrondo's paradox. The implications of this result are then discussed in the concluding section.

2. Games and winning condition

Parrondo's paradox can be illustrated by two coin-flipping games, A and B. In game A, a player flips a coin with the probability of obtaining heads as 0.495. Every head obtained increases the capital by 1, and every tail reduces the capital by 1. Game B is a capital dependent game, where if the capital is a multiple of 3, then the player flips a coin with the probability of obtaining heads as 0.095. Otherwise, he flips a coin with the probability of obtaining heads as 0.745. These games are stochastic processes and may be expressed as stochastic matrices,

$$\mathbf{P}_A = \begin{pmatrix} 0 & 0.495 & 0.505 \\ 0.505 & 0 & 0.495 \\ 0.495 & 0.505 & 0 \end{pmatrix}, \tag{1}$$

$$\mathbf{P}_B = \begin{pmatrix} 0 & 0.095 & 0.905 \\ 0.255 & 0 & 0.745 \\ 0.745 & 0.255 & 0 \end{pmatrix}. \tag{2}$$

We provide a general form of these games by representing them as the following stochastic matrix,

$$\mathbf{P} = \begin{pmatrix} 0 & p_0 & 1-p_0 \\ 1-p_1 & 0 & p_1 \\ p_2 & 1-p_2 & 0 \end{pmatrix}. \tag{3}$$

To obtain the net capital gain per game, we first compute the stationary probability vector $\boldsymbol{\pi} = (\pi_0, \pi_1, \pi_2)$ from the equation

$$\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}, \tag{4}$$

and the net capital gain would take the following expression,

$$E(X) = \sum_{i=0}^{2} \pi_i p_i - \sum_{i=0}^{2} \pi_i (1-p_i), \tag{5}$$

where X is the player's gain. To obtain the condition for zero net capital gain, we solve for $E(X) = 0$ and find that

$$\prod_{i=0}^{2} \frac{p_i}{1-p_i} = 1. \tag{6}$$

When the expression is greater than 1, the net gain is positive.

3. Complementary process

In the coin-flipping example, playing games A and B in a random order would produce a net gain in capital, while playing them on their own would give a net loss in capital. The stochastic matrix **R** formed from playing **P** and **Q** in a random order is given by

$$\mathbf{R} = \frac{1}{2}(\mathbf{P} + \mathbf{Q}). \tag{7}$$

Using the conditions that the initial two processes must be losing and the random mix must be winning, we find that in order for a process **P** to have a chance of achieving Parrondo's paradox, there must exist a complementary process **Q** such that

$$\prod_{i=0}^{2} \frac{p_i + q_i}{2 - p_i - q_i} > 1. \tag{8}$$

In other words, we can try to maximize (8) with respect to the parameters of **Q** (under the constraint that it must give a net loss in capital), and

- if the maximum value is greater than 1, the process can achieve Parrondo's paradox with the correct complementary process;
- if the maximum value is 1, the process can become fair by randomly mixing with the correct complementary process;
- if the maximum value is less than 1, the process cannot achieve Parrondo's paradox.

4. Theoretical consideration

Theorem 1. *A Parrondo game with stochastic matrices of the form (3) has a complementary process if and only if*

$$p_2 > \frac{2(1-p_0)(1-p_1)}{(1-p_0)(1-p_1)+(1+p_0)(1+p_1)}, \tag{9}$$

which is obtained by maximizing (8). Note that we may replace $\{p_0, p_1, p_2\}$ with $\{p_1, p_2, p_0\}$ or $\{p_2, p_0, p_1\}$, and only one case has to be true.

Proof. ($\Rightarrow$) Recall that the complementary process must be a losing process. That is, if the parameters for the complementary process are given by $\{q_0, q_1, q_2\}$, then

$$q_2 < \frac{(1-q_0)(1-q_1)}{q_0q_1+(1-q_0)(1-q_1)}. \tag{10}$$

Note that this inequality is the rewritten form of (6), with the product set to be less than 1. It is without loss of generality; we may interchangeably express q_0 or q_1 as the subject.

We will attempt to maximize (8) under this constraint. First, we see that the partial derivative (with respect to q_2) expression is always positive regardless of their values. This suggests that we should increase q_2 to its maximum value, that is,

$$q_2 = \lim_{\epsilon\to 0} \frac{(1-q_0)(1-q_1)}{q_0q_1+(1-q_0)(1-q_1)} - \epsilon. \tag{11}$$

We will substitute this expression into (8), and obtain

$$\frac{(p_0+q_0)(p_1+q_1)(2p_2q_0q_1-p_2q_0-p_2q_1+p_2+q_0q_1-q_0-q_1+1)}{(2-p_0-q_0)(2-p_1-q_1)(-2p_2q_0q_1+p_2q_0+p_2q_1-p_2+3q_0q_1-q_0-q_1+1)}. \tag{12}$$

It can be checked that taking the partial derivatives with respect to q_0 and q_1 individually will yield no solution for obtaining maxima in the range of $0 \le q_0 \le 1$ and $0 \le q_1 \le 1$, respectively. Therefore we can just verify the corner points. They are $(0, 0)$, $(0, 1)$, $(1, 0)$ and $(1, 1)$, in the form of (q_0, q_1):

$$\lim_{(q_0,q_1)\to(0,0)} Y = \frac{p_0p_1(1+p_2)}{(2-p_0)(2-p_1)(1-p_2)} \tag{13}$$

$$\lim_{(q_0,q_1)\to(0,1)} Y = \frac{p_0(1+p_1)(1+p_2)}{(2-p_0)(1-p_1)(1-p_2)} \tag{14}$$

$$\lim_{(q_0,q_1)\to(1,0)} Y = \frac{p_1(1+p_0)(1+p_2)}{(1-p_0)(2-p_1)(1-p_2)} \tag{15}$$

$$\lim_{(q_0,q_1)\to(1,1)} Y = \frac{p_2(1+p_0)(1+p_1)}{(1-p_0)(1-p_1)(2-p_2)} \tag{16}$$

where Y is the expression in (12). By inspection, we see that (13) is the smallest, while (14)–(16) are cyclic. By a pairwise comparison, we see that (14) is the largest if

$$\frac{(1+p_0)(2-p_0)}{p_0(1-p_0)} < \frac{(1+p_1)(2-p_1)}{p_1(1-p_1)} \tag{17}$$

$$\frac{(1+p_0)(2-p_0)}{p_0(1-p_0)} < \frac{(1+p_2)(2-p_2)}{p_2(1-p_2)}. \tag{18}$$

Equivalently, we may summarize (17) and (18) as

$$\min\left(\frac{(1+p_i)(2-p_i)}{p_i(1-p_i)}\right) = \frac{(1+p_0)(2-p_0)}{p_0(1-p_0)}, \tag{19}$$

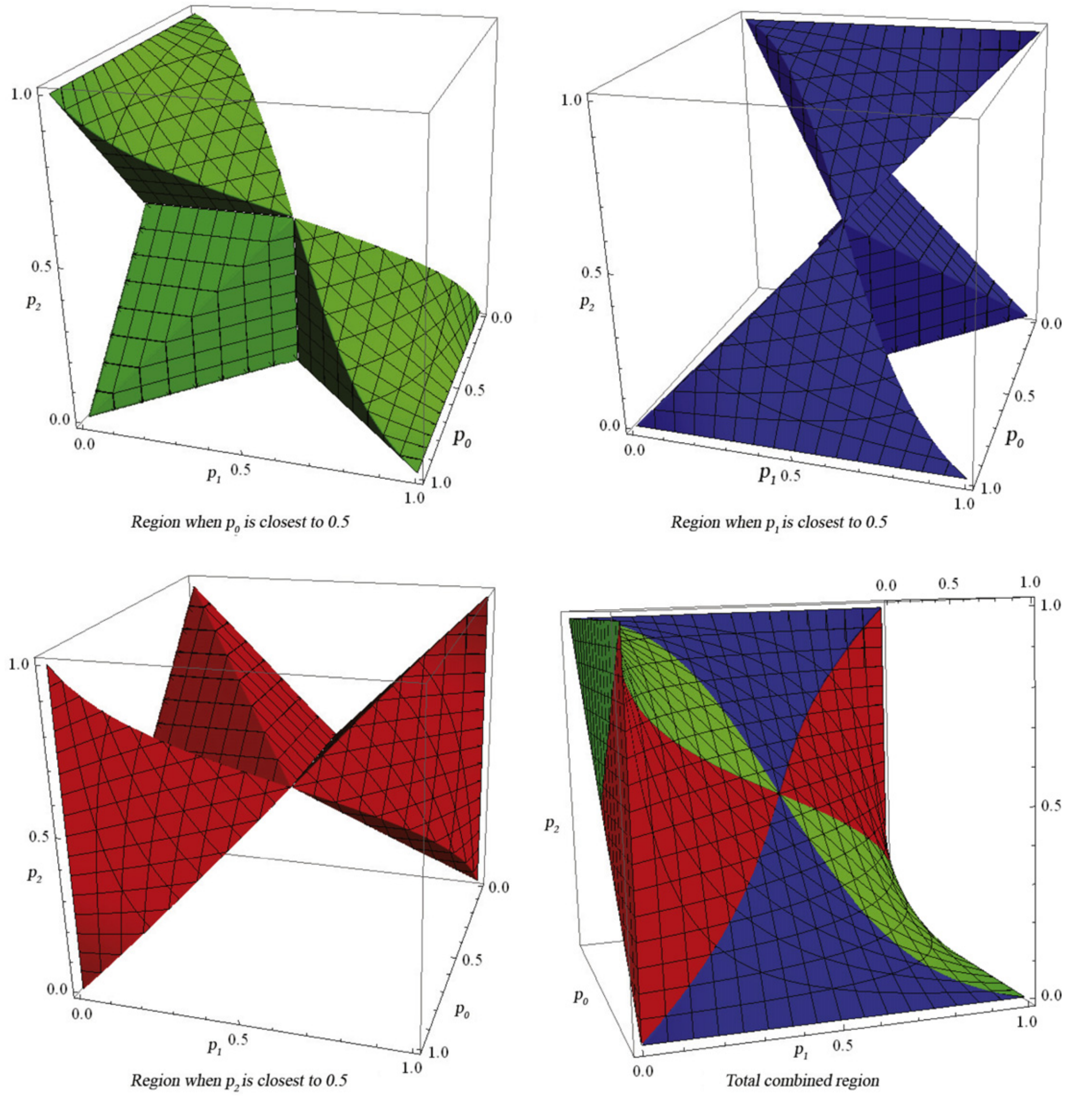

Fig. 1. (*a*), (*b*), (*c*) show the regions describing the parameters of a losing process that can have a complementary. (*d*) shows the combined region, along with the resultant boundary curve below which represents the parameters of a losing process.

where $i \in \{0, 1, 2\}$. In fact, since the function

$$f(p) = \frac{(1+p)(2-p)}{p(1-p)} \tag{20}$$

is concave up in the range of (0, 1), and has a minimum at $p = 0.5$, we may conclude that (19) is true when p_0 is the closest to 0.5. That is,

$$\min\left(\frac{(1+p_i)(2-p_i)}{p_i(1-p_i)}\right) \Rightarrow \min(|0.5 - p_i|). \tag{21}$$

Since (5) is cyclic, we may rearrange the parameters in a Parrondo process, such that p_2 in (p_0, p_1, p_2) becomes the probabilistic value closest to 0.5. For example, if p_1 is the term closest to 0.5 in the set of parameters (p_0, p_1, p_2), we may rearrange them such that it becomes (p_2, p_0, p_1), so that p_1 becomes the new p_2' in (p_0', p_1', p_2'). We must preserve the cyclic sequence $p_0, p_1, p_2, p_0, p_1, p_2, \ldots$ in order to allow (16) to be true without loss of generality. Finally, we obtain the value of q_2:

$$q_2 = \lim_{(q_0,q_1)\to(1,1)} \frac{(1-q_0)(1-q_1)}{q_0q_1 + (1-q_0)(1-q_1)} = 0. \tag{22}$$

We then substitute $(q_0, q_1, q_2) = (1, 1, 0)$ into (8) and obtain the desired inequality.

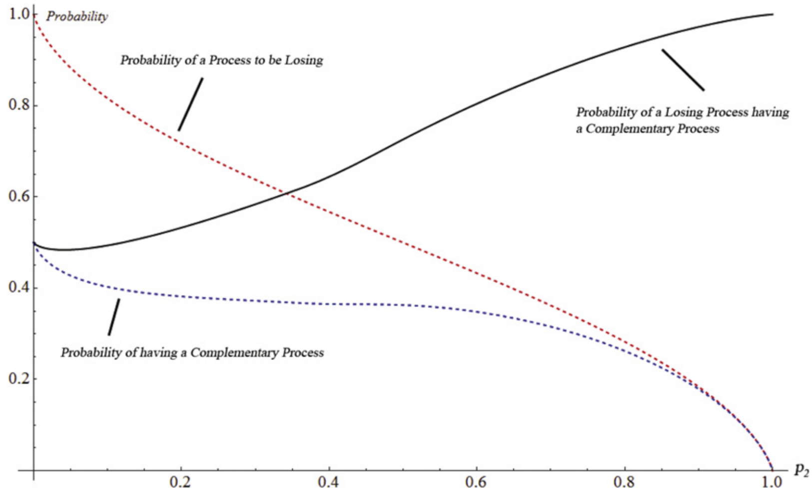

Fig. 2. p_2 represents the parameter of a process whose value is closest to 0.5. The red line represents the probability of having a losing process, and the blue line represents the probability of having a losing process with a complementary. The black line shows the probability of a losing process having a complementary, and is found to be consistently greater than 0.5. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

($\Leftarrow$) Suppose (9) is true. We have shown that this expression is obtained by two losing Parrondo processes: the first one with parameters (p_0, p_1, p_2), and the second one with parameters $(1 - \epsilon, 1 - \epsilon, 0)$, both of which are losing. Hence the first process must have a complementary process. □

5. Parameter space

By considering the condition required for a process to achieve Parrondo's paradox, we can effectively maximize the parameter space. Given three arbitrary probabilistic values, we may use (9) to easily check if it has a complementary process. We assume without loss of generality that p_2 is closest to 0.5; hence we have to consider three cases, where p_0, p_1 or p_2 is the closest to 0.5. Since at least one of the three cases must be true, the sum of parameter spaces of the three cases must fill up a unit cube.

Implementing the condition in (9) on every case separately and combining the parameter spaces, we obtain the region which accounts for every single stochastic process with a matrix of the form in (3) that is able to achieve Parrondo's paradox. The three separate regions, as well as their combined volume, are summarized in Fig. 1.

Numerical simulation predicts that 64.9...% of the volume under the boundary curve (below which represents the parameters of losing processes) is covered. This means that around two-thirds of losing processes can have a complementary process and are therefore capable of achieving Parrondo's paradox. It is important to highlight this value because the counterintuitive paradox has always been thought to be of rare occurrence. There are potentially new possibilities of discovering the phenomenon in various fields, where the chances of it being achieved have once been thought to be very low. While the investigation considers processes with stochastic matrices of the form in (3), the idea and method of investigation can be generalized to consider other kinds of stochastic matrices.

In our next investigation, we let p_2 be the probabilistic value closest to 0.5 without loss of generality. We consider how this value affects the chances of a process to have a complementary. Similarly, this is simulated by using the result in (9). Fig. 2 summarizes the results.

The probability of a process having a net capital loss (as depicted by the red dashed line) starts off from 1 and goes to 0 as p_2 increases. This is expected because when $p_2 = 0$, the process cannot be winning. Likewise, when $p_2 = 1$, the process cannot be losing. The blue dashed line represents the probability of having a process with a net loss, but has a complementary process to achieve Parrondo's paradox. It starts from 0.5 and slowly decreases to 0. The solid line represents the probability of a losing process to have a complementary, and is also the quotient of the dotted graphs. Numerical simulation predicts that the probability starts off from 0.5, drops to 0.4839... when $p_2 = 0.04119\ldots$, and then increases to 1. The mathematical explanation behind this is that the probability of finding a losing process decreases slower than the probability of finding a losing process that has a complementary initially, but decreases faster afterwards.

The simulated result also predicts that regardless of the value of p_2, the probability of a losing process to have a complementary process is more than 0.5 most of the time. Recall that the meaning behind p_2 is just how close the parameter values are to 0.5 and is therefore an approximate gauge of the values of all three parameters. This implies that regardless of the values of all three parameters, the probability of a losing process being able to have a complementary is high, further strengthening the idea that Parrondo's paradox is rather common in this family of stochastic matrices.

6. Conclusion

We have presented the conditions required by two processes in order to achieve Parrondo's paradox, and summarize all conditions into one single inequality, in (9). Based on this condition, we determine the probability of a losing Parrondo process to be able to achieve the paradox. This value is found to be a value greater than 0.5, and is of considerable significance. Contrary to common intuition that the paradox is a special case arising from a specific set of numbers, our results have shown that this may not be the case after all. Processes that can achieve Parrondo's paradox can be quite common, and we are able to introduce complementary processes for them, as seen in our analysis. However, this does not mean that searching for such a process is an easy task; we have only shown that it is not as difficult as it was previously thought.

Furthermore, there is no specific set of values required for the complementary process, as seen in our analysis which makes no assumptions about the complementary process. One may thus choose freely from every available complementary process, and select the most optimal process based on additional restrictions in real life. The next investigation shows that regardless of the values of the parameters, the probability of a process having a complementary is generally more than 0.5 as well. In addition, our analysis using stochastic matrices may be generalized to consider other stochastic processes as well. There is also no restriction on it being time discrete, since a time continuous variant can be considered accordingly.

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