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Parrondo's paradox and complementary Parrondo processes

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ABSTRACT

Parrondo's Paradox has gained a fair amount of attention due to it being counter-intuitive. Given two stochastic processes, both of which are losing in nature, it is possible to have an overall net increase in capital by periodically or randomly alternating between the two processes. In this paper, we analyze the paradox with a different approach, in which we start with one process and seek to derive its complementary process. We will also state the conditions required for this to occur. Possible applications of our results include the development of future models based on the paradox.

1. Introduction

In the example provided by Parrondo [1,2], we consider two coin-flipping games A and B. We start off with 0 points, and for every head flipped, we gain 1 point, while for every tail flipped, we lose 1 point. In game A, we flip a coin with probability of obtaining heads $p_A = 0.495$. In game B, we first consider our capital. If our capital is a multiple of 3, we flip a coin with probability of obtaining heads $p_{B1} = 0.095$. Otherwise, we flip a coin with probability of obtaining heads $p_{B2} = 0.745$. This game configuration was first analyzed using discrete time Markov chains [3]. The stochastic matrices $\mathbf{P}_A$ and $\mathbf{P}_B$ for games A and B respectively, are as follows

$$\mathbf{P}_A = \begin{pmatrix} 0 & 0.495 & 0.505 \\ 0.505 & 0 & 0.495 \\ 0.495 & 0.505 & 0 \end{pmatrix} \tag{1}$$

$$\mathbf{P}_B = \begin{pmatrix} 0 & 0.095 & 0.905 \\ 0.255 & 0 & 0.745 \\ 0.745 & 0.255 & 0 \end{pmatrix}. \tag{2}$$

It is shown that when playing the two games on their own, both will tend to have a net decrease in capital. However, when they are played in a random order, or in certain periodic alternation, the net capital actually shows an increasing trend [1].

One explanation for this counterintuitive phenomena is provided by Toral [4], where he considers how one losing game may alter the distribution of capital, such that it becomes favorable for the capital dependent game.

In Ref. [5], it is found that a greedy strategy, where a player gets to choose which game to play after considering its current capital, will ultimately be a losing strategy.

Since its discovery, many applications have been found for the paradox [6]. An application of the paradox may be found in the consideration of quantum models [7,8], where a direct analogy between probabilistic variables in the paradox is

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related to quantum phenomena. In fact, the discovery of Parrondo's paradox was attributed to the study of the Brownian Ratchet [9]. One of the earlier studies in the application of Parrondo's paradox to quantum models is by Meyer et al. [10], where a quantum lattice gas is used to consider a Parrondo game in the quantum sphere.

Other applications include population modelling [11–13]. In Ref. [11], cells take up discrete cell states, and undergo a sequence of stochastic processes. It is shown that in certain situations, random switching of cell phenotypes would allow for better fitness and thus promote population growth.

Parrondo's paradox may also be found in controlling chaos [14,15]. In Ref. [14], Almeida et al. considers how two chaotic systems can give rise to order, in the form of quadratic maps. This is then related to Parrondo's paradox, due to its similarity in terms of having a *lose* + *lose* = *win* situation.

The paradox has also been considered in reliability theory [16]. Starting with units of a system being less reliable than the units of another system, Crescenzo has shown that by randomly choosing units from these two systems, it is possible to obtain a system that is more reliable than the initial two.

In many cases, the consideration of the paradox is how two processes will mix and create the phenomenon. We consider a different approach, where we are already given a process which will produce a negative expected value. We aim to derive the conditions required for the process to have a complementary process, such that when these two processes are switched randomly or periodically, they produce positive returns.

Some models [7,8,11,14] consider achieving the paradox by constructing two processes as an analogue to Parrondo games. Such constructions do produce the counterintuitive result, but impose conditions on both processes. In our paper, we first consider a single process and derive the conditions required for it to be eligible as a process achieving Parrondo's Paradox. The complementary process, if found to exist, can then be strategically selected such that Parrando's Paradox can take place. This has potential applications in the development of quantum models, as well as other fields where the paradox can be found.

2. Definitions

Definition 1. We define a Parrondo process to be a process with a stochastic matrix of the form

$$\mathbf{P} = \begin{pmatrix} 0 & p_0 & 1-p_0 \\ 1-p_1 & 0 & p_1 \\ p_2 & 1-p_2 & 0 \end{pmatrix}. \tag{3}$$

The coin-flipping example will thus have the following stochastic matrices $\mathbf{P}_A$ and $\mathbf{P}_B$ for games A and B respectively,

$$\mathbf{P}_A = \begin{pmatrix} 0 & 0.495 & 0.505 \\ 0.505 & 0 & 0.495 \\ 0.495 & 0.505 & 0 \end{pmatrix}, \tag{4}$$

$$\mathbf{P}_B = \begin{pmatrix} 0 & 0.095 & 0.905 \\ 0.255 & 0 & 0.745 \\ 0.745 & 0.255 & 0 \end{pmatrix}. \tag{5}$$

Definition 2. The expected value of a Parrondo process is defined as the overall probability of transiting from state i to state $i+1 (\mathrm{mod}\ n)$ minus the overall probability of transiting from state i to $i-1 (\mathrm{mod}\ n)$, $(i = 1, 2, 3 \ldots n, n \in \mathbb{N})$.

In order to compute the expected value of a process, we first find the stationary probability vector of its stochastic matrix. In game A of the coin-flipping example, we find the stationary probability vector $\boldsymbol{\pi}_A$ to be the following

$$\boldsymbol{\pi}_A = (0.333 \quad 0.333 \quad 0.333). \tag{6}$$

Therefore if we set X to be the random variable associated with the Parrondo process, the overall probability of transiting from state i to state $i+1 (\mathrm{mod}\ 3) (i \in 0, 1, 2)$ (which we will define to be $X = 1$) takes the following expression,

$$P(X = 1) = \sum_{i=0}^{2} \pi_i p_i. \tag{7}$$

Note that π_i is the ith entry in the stationary probability vector, while p_i represents the probability of transiting from state i to state $i+1 (\mathrm{mod}\ 3)$. Similarly, we have the overall probability of transiting from state i to state $i-1 (\mathrm{mod}\ 3) (i \in 0, 1, 2)$ (taken to be $X = -1$),

$$P(X = -1) = \sum_{i=0}^{2} \pi_i (1 - p_i). \tag{8}$$

Thus, the expected value of a Parrondo process is given by

$$E(X) = \sum_{i=0}^{2} \pi_i p_i - \sum_{i=0}^{2} \pi_i (1 - p_i). \tag{9}$$

For game A in the coin-flipping example, $E_A(X) = -0.01$, while for game B, $E_B(X) = -0.0087$.

Definition 3. A Parrondo process is said to be winning if its expected value is positive. It is losing if its expected value is negative, and fair if the expected value is zero.

Thus both games A and B are losing, since their expected values are both negative.

Definition 4. Consider a losing process X. Process Y is said to be a (x, y) complementary process for process X if process Y is also losing, but when these two processes are performed in a random order in a proportion of x to y $(x + y = 1; x, y \geq 0)$, the overall combination becomes winning.

Lemma 1. *If a losing process X has a (x, y) complementary process Y, then process X is a (y, x) complementary process for process Y.*

Proof. We start with a losing process X, with a complementary process Y. In that case, Y has to be losing. Also, the random mix of the two processes will become winning. The stochastic matrix of this resultant winning matrix is given by the weighted mean of the two individual stochastic matrices, in their respective proportions

$$\mathbf{P}_{XY} = x\mathbf{P}_X + y\mathbf{P}_Y. \tag{10}$$

Now we start with process Y, which is also losing. Process X is a losing process, and when randomly mixed with process Y in a ratio of y to x, will give a process with the following resultant stochastic matrix

$$\mathbf{P}_{YX} = y\mathbf{P}_Y + x\mathbf{P}_X. \tag{11}$$

This process has already been found to be winning from (10), and hence process X satisfies every condition to be a (y, x) complementary process for Y as well. □

The resultant matrix from a random, equal mix of games A and B will be the following,

$$\frac{1}{2}\mathbf{P}_A + \frac{1}{2}\mathbf{P}_B = \begin{pmatrix} 0 & 0.295 & 0.705 \\ 0.380 & 0 & 0.620 \\ 0.620 & 0.380 & 0 \end{pmatrix}. \tag{12}$$

The expected value for this process is 0.0157, which is positive. Thus game B is a $(0.5, 0.5)$ complementary process for game A, and vice versa. This counterintuitive result forms the motivation for our investigation. We aim to derive a complementary process for a given Parrondo process.

3. Conditions for winning

We first find the general stationary probability vector for a stochastic matrix as in (3)

$$\boldsymbol{\pi} = \begin{pmatrix} \pi_0 & \pi_1 & \pi_2 \end{pmatrix}, \tag{13}$$

where:

$$\pi_0 = -\frac{-p_2 p_1 + p_1 - 1}{p_0 p_1 + p_0 p_2 + p_1 p_2 - p_0 - p_1 - p_2 + 3}, \tag{14}$$

$$\pi_1 = -\frac{-p_2 p_0 + p_2 - 1}{p_0 p_1 + p_0 p_2 + p_1 p_2 - p_0 - p_1 - p_2 + 3}, \tag{15}$$

$$\pi_2 = -\frac{-p_0 p_1 + p_0 - 1}{p_0 p_1 + p_0 p_2 + p_1 p_2 - p_0 - p_1 - p_2 + 3}. \tag{16}$$

Using this general solution, we equate the expected value, found using (9), to zero. Simplifying the expression, we have

$$p_0 = \frac{p_1 p_2 - p_1 - p_2 + 1}{2p_1 p_2 - p_1 - p_2 + 1}. \tag{17}$$

We will define $f(p_1, p_2)$ to be

$$f(p_1, p_2) = \frac{p_1 p_2 - p_1 - p_2 + 1}{2p_1 p_2 - p_1 - p_2 + 1}. \tag{18}$$

It is checked that when $p_0 > f(p_1, p_2)$, the process is winning. Hence, we may restate our initial problem as the following:

Beginning with a Parrondo process with parameters p_0, p_1, p_2, where $p_0 < f(p_1, p_2)$, we aim to derive another process with parameters q_0, q_1, q_2, where $q_0 < f(q_1, q_2)$, such that for some $0 \le k \le 1$, we have

$$(1-k)p_0 + kq_0 > f((1-k)p_1 + kq_1, (1-k)p_2 + kq_2). \tag{19}$$

We will let $r_i = (1-k)p_i + kq_i$, $i \in \{0, 1, 2\}$. Since r_i is a weighted proportion between p_i and q_i, we know that

$$\min(p_i, q_i) \le r_i \le \max(p_i, q_i). \tag{20}$$

Without loss of generality we let $p_i \le q_i$, so that $p_i \le r_i \le q_i$.

4. Extrema and their significance

We let

$$p_1 + p_2 = c. \tag{21}$$

Since c is the sum of two variables, it is also a variable. Hence we may rewrite f as

$$f(c, p_1) = \frac{-p_1^2 + cp_1 - c + 1}{-2p_1^2 + 2cp_1 - c + 1}. \tag{22}$$

Since $p_0 < f(c, p_1)$ and $r_0 > f(c, r_1)$, we know that there definitely exists some proportion of q_i and p_i, say r_i', such that

$$r_0' = f(c, r_1'). \tag{23}$$

In order to study this, we define another function

$$g(c, p_0, p_1) = f(c, p_1) - p_0. \tag{24}$$

Since p_0 and p_1 are not independent of each other, we can simplify the function to be

$$g(c, x) = f(c, p_1 + (q_1 - p_1)x) - (p_0 + (q_0 - p_0)x) \tag{25}$$

where q_0, q_1, p_0, p_1 are all constants. Note that x describes the proportion between the original process and the complementary process.

Theorem 1. *It is only possible to construct a complementary process under the sufficient condition that $c < 1$.*

Proof. We first consider $c = 1$., This means that $f(1, p_1) = 0.5$ regardless of the value of p_1. In order for there to be a complementary process, we have the conditions:

$$p_0 < f(1, p_1) = 0.5, \tag{26}$$
$$q_0 < f(1, q_1) = 0.5, \tag{27}$$
$$r_0 > f(1, r_1) = 0.5. \tag{28}$$

But since $r_0 < q_0$, we have

$$q_0 > r_0 > 0.5, \tag{29}$$

which contradicts (27). Now we consider $c \ne 1$. We first consider the second derivative of g. Note that since $(p_0 + (q_0 - p_0)x)$ is a linear function, it will not have a second derivative. Hence we only consider f. It is found that there is only one value of x for which g is an extrema, given by

$$x = \frac{c - 2p_1}{2(q_1 - p_1)}. \tag{30}$$

At this point, the second derivative of g is given by

$$\left.\frac{\partial^2 g}{\partial x^2}\right|_{x=\frac{c-2p_1}{2(q_1-p_1)}} = \frac{(1-c)(q_1 - p_1)^2}{c^2 - 2c + 2}. \tag{31}$$

This means that for values of c greater than 1,

$$\left.\frac{\partial^2 g}{\partial x^2}\right|_{x=\frac{c-2p_1}{2(q_1-p_1)}} < 0. \tag{32}$$

It is then checked that for other values of x, the sign of the concavity remains the same. We return to the conditions required for a complementary process to be constructed. For $c > 1$,

$$g(c, p_0, p_1) > 0, \tag{33}$$

$$g(c, q_0, q_1) > 0, \tag{34}$$

$$g(c, r_0, r_1) < 0. \tag{35}$$

Consider $p_i \le r_i' \le r_i$, $i \in \{0, 1, 2\}$. Since g has changed from positive to negative, it must have a negative gradient at some point. Due to the fact that concavity is negative as well, this negative gradient will increase in magnitude. This means that for $r_i \le r_i'' \le q_i$, the gradient will always be negative, and thus it is impossible for g to rise back up to a positive value. Hence for $c > 1$ it is impossible for a complementary process to be constructed.

However, when $c < 1$,

$$\left.\frac{\partial^2 g}{\partial x^2}\right|_{x=\frac{c-2p_1}{2(q_1-p_1)}} > 0. \tag{36}$$

Again it is checked that throughout the values of x the sign of the concavity remains the same. The conditions for a complementary process to be generated are

$$g(c, 0) > 0, \tag{37}$$

$$g(c, 1) > 0, \tag{38}$$

$$g(c, k) < 0. \tag{39}$$

We will find an example to illustrate its possibility. We will start with the condition $p_0 = q_0 = r_0$. Suppose we set the conditions

$$\frac{(2-c)^2}{2c^2-4c+4} < p_0 < f(c, p_1), \tag{40}$$

$$p_1 < \frac{c}{2}. \tag{41}$$

Note that the expression

$$\frac{(2-c)^2}{2c^2-4c+4}, \tag{42}$$

is the minimum of f, and therefore we can always find p_0 for (40). And from (40) we satisfy our first condition in (37). Then we let

$$k = \frac{c-2p_1}{2(q_1-p_1)}. \tag{43}$$

Recall that for this value of k, f is at its minimum. Therefore,

$$g(c, k) = \frac{(2-c)^2}{2c^2-4c+4} - r_0 < 0. \tag{44}$$

The last inequality follows from (40) and the fact that $r_0 = p_0$. Now, we consider f. It can be checked that f is symmetric about $p_1 = c/2$, implying that if we let

$$q_1 = c - p_1, \tag{45}$$

we will satisfy (38). Therefore we are done. □

Theorem 1 is proven only if $p_1 + p_2 = q_1 + q_2 = c$. Hence the condition is sufficient but not necessary. Nonetheless, we will show that no processes are limited by this condition in Proposition 1.

We will now move on to deriving every losing process that satisfies this condition.

5. Existence of complementary process

While a number of processes do not satisfy the condition as described in Theorem 1, we may modify these existing processes to satisfy the condition. Suppose we start with a losing Parrondo process that does not satisfy the condition

$$\mathbf{P} = \begin{pmatrix} 0 & p_0 & 1-p_0 \\ 1-p_1 & 0 & p_1 \\ p_2 & 1-p_2 & 0 \end{pmatrix}, \tag{46}$$

$$p_1 + p_2 \ge 1. \tag{47}$$

By the symmetry of Parrondo processes, we may rearrange the states (say states 0 and 2) to obtain the matrix

$$\mathbf{P} = \begin{pmatrix} p_2 & 1-p_2 & 0 \\ 1-p_1 & 0 & p_1 \\ 0 & p_0 & 1-p_0 \end{pmatrix}, \tag{48}$$

such that now we require the condition

$$p_0 + p_1 < 1, \tag{49}$$

to be satisfied instead.

Proposition 1. *Every losing process satisfies the sufficient condition described in Theorem 1.*

Proof. For a losing process with parameters p_0, p_1, p_2, we have

$$p_0 < \frac{p_1p_2 - p_1 - p_2 + 1}{2p_1p_2 - p_1 - p_2 + 1}, \tag{50}$$

and therefore it suffices to prove that one of the three inequalities below:

$$p_1 + p_2 < 1, \tag{51}$$

$$p_1 + \frac{p_1p_2 - p_1 - p_2 + 1}{2p_1p_2 - p_1 - p_2 + 1} < 1, \tag{52}$$

$$p_2 + \frac{p_1p_2 - p_1 - p_2 + 1}{2p_1p_2 - p_1 - p_2 + 1} < 1, \tag{53}$$

is satisfied. If $p_1 + p_2 < 1$, we are immediately done. If it is not satisfied, then at least one of them, say p_i, satisfies the inequality

$$p_i > 0.5. \tag{54}$$

We will denote the other parameter as p_j. Now we consider

$$p_j + \frac{p_jp_i - p_j - p_i + 1}{2p_jp_i - p_j - p_i + 1} = \frac{p_j^2(2p_i - 1) - p_i + 1}{p_j(2p_i - 1) - p_i + 1}. \tag{55}$$

Since $(2p_i - 1)$ is positive, we can conclude that the expression is less than 1. If both p_1, p_2 are greater than 0.5, we may still use the same argument. And hence we are done. □

The significance of this result lies in the fact that the condition does not eliminate any losing process from having a complementary process. We will next find a lower bound for c.

Theorem 2. *Let a Parrondo process and its complementary process have parameters p_0, p_1, p_2 and q_0, q_1, q_2 respectively, where*

$$p_1 + p_2 = q_1 + q_2 = c, \tag{56}$$

and

$$p_0 > q_0. \tag{57}$$

If we define q_1' such that

$$q_1' = p_1 + \frac{p_0(q_1 - p_1)}{p_0 - q_0}, \tag{58}$$

then a process with the following stochastic matrix

$$\mathbf{P} = \begin{pmatrix} 0 & 0 & 1 \\ 1-q_1' & 0 & q_1' \\ c-q_1' & 1-(c-q_1') & 0 \end{pmatrix}, \tag{59}$$

is also a complementary process of the original Parrondo process, if the values are within the range for a well-defined process.

Proof. Suppose the resultant winning process formed by p_0, p_1, p_2 and q_0, q_1, q_2 satisfies

$$g(c, k_0) < 0, \tag{60}$$

for some $k = k_0$. This means that when $k = k_0$, we obtain a weighted proportion of the Parrondo process and its complementary such that the resultant process is winning. We also know that if $k = 1$, we will obtain the complementary process since it is in full proportion of the complementary process. However, if we further increase the value of k, say,

$$k = \frac{p_0}{p_0 - q_0}, \tag{61}$$

then we will obtain the process with stochastic matrix in (59). Note that the complementary process is in proportions greater than 1, while the original Parrondo process is in negative proportions.

We now only consider the original Parrondo process with parameters p_0, p_1, p_2 and a new process with parameters $0, q_1', c - q_1'$. This new process is a losing process, because

$$g(c, k) > 0, \tag{62}$$

for $k \geq 1$, the reason being that $g(c, 1) > 0$ and the second derivative of g with respect to k is positive. With this new process, we can now define a k' as the weighted proportion between the original process and the new process, as in $g(c, k')$.

$$\frac{p_0 - q_0}{p_0} k = k'. \tag{63}$$

This will mean that when $k' = 1$, we obtain the new process. In order to obtain the winning process, we have

$$k_0' = \frac{p_0 - q_0}{p_0} k_0, \tag{64}$$

and therefore there exists a k_0' such that

$$g(c, k_0') = g(c, k_0) < 0. \tag{65}$$

We have shown that the new process is losing, and that there is some k_0' such that the process is winning. Thus this new process is a complementary process of the original Parrondo process. □

Theorem 3. *If a losing Parrondo process has parameters p_0, p_1, p_2 such that*

$$\sqrt{3} - 1 < p_1 + p_2, \tag{66}$$

then it is always possible to derive a complementary process with parameters q_0, q_1, q_2 such that

$$q_1 + q_2 = p_1 + p_2 = c. \tag{67}$$

Proof. We define q_1' as in Theorem 2. We also introduce a new process with parameters $\{f_0, f_1, f_2\}$. This process is fair and is formed from the linear combination of the original process and the process defined in (59). Since there exists some linear combination which results in a winning process, with the initial two processes being losing, there will definitely be two such fair processes. We select the process that has a greater proportion of the process defined in (59) than the winning process. Now, we consider a linear combination,

$$(1 - \kappa)\{f_0, f_1, f_2\} + \kappa\{0, q_1', c - q_1'\}. \tag{68}$$

By the same argument in Theorem 2, it can be shown that the corresponding k_0' for such processes is given by

$$k_0' = \frac{p_0 - q_0}{p_0 - (1 - \kappa) f_0}. \tag{69}$$

If $\kappa > 1$, we see that the first parameter becomes negative, and therefore κ is not well-defined at this range. If $\kappa < 0$, then the process we obtain would be winning, and would thus violate the condition for it to be a complementary process. Hence, this linear combination results in another complementary process, for $0 \leq \kappa \leq 1$. Since we have considered every possible value for κ, we have found every possible complementary process for a given original process and its corresponding process involving q_1'.

Finally, we note that the process defined in (59) may not always exist, where it may be greater than 1 in some cases. We should, in that case, consider only the well-defined processes formed from the linear combination. This way, we extend the range of q_1' to be

$$-\infty < q_1' < \infty. \tag{70}$$

It now suffices to show that for all values of q_1', complementary processes may be generated. This statement, in addition to the fact that we have found every complementary process for a given value of q_1', would thus allow us to obtain every complementary process as long as it satisfies (67).

We now wish to find the range of q_1' for a given process. It can be checked that the limiting case occurs when the complementary process can only make the process fair, and not winning. Also, if we start with a fair process, and set $k = 0$ for the linear combination of the process and its complementary, we see that we will get the same fair process. Hence, the limiting case occurs when we start with a fair process. Now we consider the value for q_1' in such a case.

Since q_1' is obtained from a linear combination of an original process and its complementary, it must be of the form

$$q_1' = p_1 + \frac{p_0}{m}, \tag{71}$$

where m depends on the choice of complementary process. For the complementary process to only make the original process fair, we find that,

$$m_0 = \frac{df(c, p_1)}{dp_1}. \tag{72}$$

This is because if $m > m_0$, then starting from the fair process, we consider a process with parameters $\{p_0 + \epsilon_0, p_1 + \epsilon_1, c - (p_1 + \epsilon_1)\}$. We first consider the equation,

$$\frac{dg(c, p_0, p_1)}{dp_1} = \frac{df(c, p_1)}{dp_1} - \frac{dp_0}{dp_1}. \tag{73}$$

For an increase ϵ_0, we find that the rate of increase of p_0 is given by m, and the first term on the right side of the equation is merely m_0. Hence, we have $g(c, p_0, p_1) < 0$ and therefore we have a winning process, which means that a complementary process is able to make a process win. If, however, $m < m_0$, we proceed with a small decrease ϵ_0, and we find that we obtain a winning process again. If $m = m_0$, we find that regardless of whether we consider an increase or decrease, we will not get winning processes. This is because the gradient of g is smaller than m_0 for a decrease, and greater than m_0 during an increase. Thus, we find that the expression for q_1' to be

$$q_1' = p_1 + \frac{p_0(2p_1^2 - 2cp_1 + c - 1)^2}{(c - 2p_1)(c - 1)}. \tag{74}$$

We now consider range at the two endpoints. When $p_1 = 0$,

$$\frac{1 - c}{c} < q_1' < \infty. \tag{75}$$

This means that for these values of q_1', we are able to generate complementary processes for a fair process with parameters $\{1, 0, c\}$. We can therefore say that the fair process is able to make the generated processes win. If we have the following parameters instead,

$$\lim_{\epsilon \to 0} \{1 - \epsilon, 0, c\}, \tag{76}$$

then we get a losing process that makes the generated processes win. Thus, we have derived a complementary process for processes generated from the range of values for q_1'. When $p_1 = c$, in the same way we find that we are able to find a complementary processes for processes generated from the range,

$$-\infty < q_1' < c - \frac{1 - c}{c}. \tag{77}$$

In order for a process with $p_1 + p_2 = c$ to always have a complementary process, the range of q_1' has to be to be from $-\infty$ to ∞,

$$\frac{1 - c}{c} < c - \frac{1 - c}{c}, \tag{78}$$

which is found to be $\sqrt{3} - 1 < c$. □

We summarize our previous results.

Corollary 1. *Any losing Parrondo process, with a stochastic matrix*

$$\mathbf{P} = \begin{pmatrix} 0 & p_0 & 1 - p_0 \\ 1 - p_1 & 0 & p_1 \\ p_2 & 1 - p_2 & 0 \end{pmatrix}, \tag{79}$$

will have a complementary process, as long as the sum c of any two of the set $\{p_0, p_1, p_2\}$ satisfies

$$\sqrt{3} - 1 < c < 1. \tag{80}$$

Proof. The result follows from Theorems 1 and 3. □

We now explore the significance of k in $g(c, k)$.

Lemma 2. *If two losing Parrondo processes satisfy*

$$g(c, 0) > 0, \tag{81}$$

$$g(c, k) < 0, \tag{82}$$

$$g(c, 1) > 1, \tag{83}$$

then one process is $(k, 1-k)$ complementary to the other.

Proof. Suppose our two processes have parameters $\{p_0, p_1, p_2\}$ and $\{q_0, q_1, q_2\}$. Then the random mix will have parameters

$$k\{p_0, p_1, p_2\} + (1-k)\{q_0, q_1, q_2\}, \tag{84}$$

which is exactly the definition for a $(k, 1-k)$ complementary process. □

6. Numerical example

We will consider an example using Parrondo's original configuration. Suppose we start with game B, with the stochastic matrix

$$\mathbf{P} = \begin{pmatrix} 0 & 0.095 & 0.905 \\ 0.255 & 0 & 0.745 \\ 0.745 & 0.255 & 0 \end{pmatrix}. \tag{85}$$

In this case we have the parameters

$$\{p_0, p_1, p_2\} = \{0.095, 0.745, 0.745\}, \tag{86}$$

and we can see that two pairwise sums satisfy

$$\sqrt{3} - 1 < 0.095 + 0.745 < 1. \tag{87}$$

Suppose we consider $p_0 + p_2 = 0.84$. Note that the process is losing because

$$p_1 = 0.745 < f(0.095, 0.745). \tag{88}$$

We then check that it has a $(0.5, 0.5)$ complementary process with parameters

$$\{p'_0, p'_1, p'_2\} = \{0.745, 0.745, 0.095\}, \tag{89}$$

where

$$p_0 + p_2 = p'_0 + p'_2. \tag{90}$$

We check that this process is also a losing process:

$$p'_1 = 0.745 < f(0.095, 0.745), \tag{91}$$

and the random mix is winning:

$$\frac{p_1}{2} + \frac{p'_1}{2} > f\left(\frac{p_0}{2} + \frac{p'_0}{2}, \frac{p_2}{2} + \frac{p'_2}{2}\right). \tag{92}$$

Thus, we are done. In other words, Theorem 3 ensures that if a losing process has parameters which satisfy (80), then it will definitely have a complementary process, with the same sum for the two probabilistic values we consider.

7. Conclusion

We have derived the conditions required for one process to have its complementary process. We consider the conditions required for a process to give a positive expected value, and study processes with a certain stochastic matrix of the form in (3).

For the values of $c < \sqrt{3} - 1$, our proof does not indicate whether it is possible to have a complementary process or not. One observation about such processes is that they have a high degree of losing—it has been proven in Theorem 1 that all losing processes must have at least 1 pairwise sum in $\{p_0, p_1, p_2\}$ whose sum is less than 1, and having this sum lower than $\sqrt{3} - 1$ will mean that the probability of losing is very high. Nevertheless, we are unable to conclude about the possibility of it having complementary processes. This motivates future work.

It has been shown that the condition required for a process of have a complementary process is the simple condition that any pairwise sum of the parameters $\{p_0, p_1, p_2\}$ is less than 1, and we proceed to show that all losing processes satisfy this condition. Therefore no losing processes have been eliminated from having a complementary process. In addition, we have shown that if this sum is greater than $\sqrt{3} - 1$, then the process will definitely have a complementary one.

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