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Construction of novel stochastic matrices for analysis of Parrondo's paradox

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HIGHLIGHTS

- A method for transforming stochastic matrices for analysis of Parrondo's paradox is proposed.
- Through this transformation, we can determine whether a process is winning or losing.
- This also allows an efficient way of introducing bias factors.

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ABSTRACT

In Parrondo's paradox, a winning strategy is formed either by playing two losing games randomly or alternating them periodically. The paradox is commonly analyzed using stochastic matrices. In this paper, we modify the stochastic matrices to allow a more systematic introduction of bias into fair processes, while retaining the use of simple matrix operations throughout the analysis.

1. Introduction

In Parrondo's paradox, playing two losing games in a random or periodic order results in a winning outcome. The original example provided by Parrondo in Ref. [1] involves playing two games, *A* and *B*, in which the player starts with zero capital. Both games involve coin-flipping, and the player's capital will increase by one for each head and decrease by one for each tail. For game *A*, the player flips a coin with a probability of obtaining heads as 0.495. For game *B*, if the player's capital is a multiple of three, then he flips a coin with a probability of obtaining heads as 0.095. Otherwise he flips a coin with a probability of obtaining heads as 0.745. When played separately, both games will produce a negative expected change in capital, but playing them in a random order will produce a positive expected change in capital [1]. It has also been shown that the paradox is apparent even in chaotic alternation between games [2,3].

Parrondo's paradox has since garnered significant attention, and is found to readily agree with further mathematical analysis. One such analysis can be found in Ref. [4], in which Toral attempts to deduce how the paradox works. Toral explains that when the player plays one of the games only, the probability distribution of the player's capital (mod 3) in the long run is disadvantageous to the player. However, introducing another game would allow the distribution of capital to change in favor of the player. The conventional approach in analyzing this paradox is to use discrete time Markov chains [5]. This method allows us to determine the steady-state behavior of the games and derive the expected change in capital when the games are played in the long run.

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Separately, Soo and Cheong have constructed a method of deriving a complementary process starting from only one process, such that Parrondo's paradox can be achieved [6].

Since its discovery, many applications have been found for the paradox [7]. Work has been carried out on its application to quantum models [8,9], where the classical coin toss is replaced by the measurement of a qubit. One of the pioneering studies in this direction is by Meyer et al. [10], where a quantum lattice gas is used to consider a Parrondo game in the quantum regime. The paradox has also been found in certain population models [11–14]. For example, cells are considered to have discrete states and random switching of these states increase its fitness [11].

More recently, an optical model providing a physical analogy of the paradox has been developed. It involves arranging beam-splitters such that photons getting reflected would 'lose' and photons passing through would 'win' [15]. A thermodynamic machine involving a stochastically driven single-level quantum dot has also been theorized, with the paradox taking place [16]. Investigations on classical Parrondo games such as the coin-flipping example are still ongoing [17].

Harmer and Abbott have compiled a review of various classical models of the paradox in Ref. [18], with emphasis on the coin-flipping example. In this paper, we have kept to a more generalized representation of that well-known coin-flipping example to demonstrate our proposed method of transforming stochastic matrices for an effective introduction of bias factors.

The coin-flipping example can be generalized into stochastic processes with three states, which is determined by a player's capital (mod 3). The stochastic matrix of game B will then be

$$\mathbf{P}_B = \begin{pmatrix} 0 & 0.095 & 0.905 \\ 0.255 & 0 & 0.745 \\ 0.745 & 0.255 & 0 \end{pmatrix}, \tag{1}$$

where the first, second and third rows represent the state where the player's capital is 0, 1, 2 (mod 3) respectively. Note that the entry in row i and column j is the transition probability from state i to state j.

In Section 2, we introduce the key definitions and conditions required for the games to be winning. In order to have an effective introduction of bias factors, it would be preferable to introduce them as multipliers. To allow this, in Section 3, we perform a transformation on the original stochastic matrix. Thereafter, we will see that the transformation is related to the conditions required for a game to be winning. Furthermore, this transformation is found to include a win over loss concept, which is commonly found in various real life situations. This is discussed in Section 4. In Section 5, we show that the conventional method for analyzing Parrondo's paradox is in no way simpler than our transformation approach. In Section 6, we consider the advantages using our method. We then provide the expression for actual expected change in capital for such games after transformation in Section 7, and also show how to combine two different stochastic matrices representing two different games in Section 8.

2. Definitions

Definition 1. A Markov process with n states is winning if the overall transition probability from state i to state $i+1$ (mod n) is higher than the overall transition probability from state i to state $i+1$ (mod n) ($i = 1, 2, 3 \ldots n$) and losing if it is lower. Otherwise it is fair.

Suppose we have a simple process that involves only coin-flipping, such as in Ref. [1]. We will define a random variable X, such that when heads is obtained, X gives a value of one, when tails is obtained, X gives a value of negative one, and zero otherwise. Then if the expected value of X is positive, the process is said to be winning. If the expected value of X is negative, then it is losing. To determine whether a process is winning or losing, we consider the general stochastic matrix $\mathbf{P}$,

$$\mathbf{P} = \begin{pmatrix} 0 & p_0 & 1-p_0 \\ 1-p_1 & 0 & p_1 \\ p_2 & 1-p_2 & 0 \end{pmatrix}, \tag{2}$$

then we find the stationary probability vector (eigenvector with eigenvalue 1)

$$\boldsymbol{\pi} = \begin{pmatrix} \pi_0 & \pi_1 & \pi_2 \end{pmatrix}, \tag{3}$$

which is found to take the following expression

$$\pi_0 = -\frac{-p_2p_1 + p_1 - 1}{p_0p_1 + p_0p_2 + p_1p_2 - p_0 - p_1 - p_2 + 3} \tag{4}$$

$$\pi_1 = -\frac{-p_2p_0 + p_2 - 1}{p_0p_1 + p_0p_2 + p_1p_2 - p_0 - p_1 - p_2 + 3} \tag{5}$$

$$\pi_2 = -\frac{-p_0p_1 + p_0 - 1}{p_0p_1 + p_0p_2 + p_1p_2 - p_0 - p_1 - p_2 + 3}. \tag{6}$$

Finally, we calculate the expected value of X

$$E(X) = \sum_{n=0}^{2} \pi_n p_n - \sum_{n=0}^{2} \pi_n (1 - p_n). \tag{7}$$

The first term on the right hand side represents the overall transition probability from state i to state $i + 1 \pmod n$, while the second term is the overall transition probability from state i to state $i + 1 \pmod n$.

Definition 2. We define a 'Parrondo configuration' as possessing two processes A and B with the stochastic matrices $\mathbf{P}_A$ and $\mathbf{P}_B$ as follows

$$\mathbf{P}_A = \begin{pmatrix} 0 & p_0 & 1-p_0 \\ 1-p_0 & 0 & p_0 \\ p_0 & 1-p_0 & 0 \end{pmatrix} \tag{8}$$

$$\mathbf{P}_B = \begin{pmatrix} 0 & p_1 & 1-p_1 \\ 1-p_2 & 0 & p_2 \\ p_2 & 1-p_2 & 0 \end{pmatrix}, \tag{9}$$

where

$$\{p_0, p_1, p_2\} = \{0.5 - \epsilon, 0.1 - \epsilon, 0.75 - \epsilon\}. \tag{10}$$

When ϵ is zero, the games are fair. This configuration (and its matrices) is well-known in the study of the paradox. We will derive the conditions required for a general process with stochastic matrices found in (2). First, we set the expected value of X to be greater than zero

$$\sum_{n=0}^{2} \pi_n p_n - \sum_{n=0}^{2} \pi_n (1 - p_n) > 0. \tag{11}$$

Solving, we get

$$\prod_{n=0}^{2} \frac{p_n}{1 - p_n} > 1, \tag{12}$$

as obtained in Ref. [18]. This is one of the important results which contributes to the motivation for the transformation of the stochastic matrix later. It is possible to tell whether a process is winning or losing simply by substituting the initial conditions into this expression.

3. Transformation of stochastic matrix

For the purpose of introducing effective bias factors, we consider a situation such that a multiplier can be easily implemented. Consider a process with the following stochastic matrix $\mathbf{P}$,

$$\mathbf{P} = \begin{pmatrix} 0 & p_0 & 1-p_0 \\ 1-p_1 & 0 & p_1 \\ p_2 & 1-p_2 & 0 \end{pmatrix}. \tag{13}$$

We now impose two conditions. The first condition is that a chance experiment only has two significant outcomes. The second condition is to have a mapping f such that when $p_1 + p_2 = 1$, we have $f(p_1) f(p_2) = 1$. We propose three relations for the map $f : x \mapsto \mathbb{R}_{\geq 0}, x \in \mathbb{R}_{\geq 0}, 0 \leq x \leq 1$ for the fulfilment of the second condition

$$f(0) = 0 \tag{14}$$

$$f(0.5) = 1 \tag{15}$$

$$\lim_{p \to 1} f(p) = +\infty. \tag{16}$$

A transformation is of the form

$$f(x) = \frac{ax + b}{cx + d}, \tag{17}$$

which takes into considerations: dilation, rotation, translation and inversion. Since it is not clear which transformation(s) are involved, it is best to consider the general transformation. Solving for the mapping, we obtain

$$f(0) = 0, \tag{18}$$

which means $b = 0$, and

$$\lim_{p\to 1} f(p) = +\infty, \tag{19}$$

which means $d = -c$. While there are 4 unknown variables to be solved for, it is known that a transformation requires three relations in order to form the mapping. With the last equation

$$f(0.5) = 1, \tag{20}$$

we find that $a = 1, b = 0, c = -1, d = 1$. While this transformation is generally used for complex numbers, it is still applicable for a simple transformation for probability (values restricted to be between zero and one).

Hence we find f to be

$$f(x) = \frac{x}{1-x}. \tag{21}$$

Note that this only holds for the case when there are only two outcomes whose probability of occurring is non-zero.

We see some similarity in the transformation as compared to the result obtained in (12). Hence we rearrange the rows of the stochastic matrix such that the entry which represents the probability of transition from state i to state $i+1 \pmod n$ $(i = 1, 2, 3 \ldots n)$ are in the diagonals. We then have

$$\mathbf{P} = \begin{pmatrix} \dfrac{p_2}{1-p_2} & \dfrac{1-p_2}{p_2} & 0 \\ 0 & \dfrac{p_0}{1-p_0} & \dfrac{1-p_0}{p_0} \\ \dfrac{1-p_1}{p_1} & 0 & \dfrac{p_1}{1-p_1} \end{pmatrix}. \tag{22}$$

This reformulation provides several key ideas which are not apparent in the original formulation. With the proposed transformation, one main advantage is that we can easily tell whether a process is winning or losing. The following lemma tells us that a simple check of the diagonals is sufficient to determine whether a process is winning or losing.

Lemma 1. *The transformed stochastic matrix in (22) represents a winning process if the product of its diagonal entries is greater than one, a losing process if it is smaller than one and a fair process if it is exactly one.*

Clearly, this is more efficient than the conventional method of deriving the expected change in capital and checking whether it is greater or lesser than zero. There are other advantages (discussed over the next few sections) which will fully justify the transformation.

4. Win over loss concept

In our transformation, we have performed an operation that involves win over loss. That is, if x is the chance of winning, then $1 - x$ will be the chance of losing. The transformation is then

$$x \to \frac{x}{1-x}. \tag{23}$$

In addition, after the transformation, our probabilistic value now ranges from zero to infinity, instead of just from zero to one. This provides added flexibility for us to formulate real life models. Let us consider a simple example to illustrate this point. Suppose we have 4 red marbles and 3 blue marbles placed in a bag. The probability of randomly selecting a red marble from the bag will be $4/(3+4)$. For the case of the transformed probabilistic value, the perceived 'probability' will be $4/3$, where the number of red balls is 4 and the number of balls that are not red is 3. Note that it is possible for the value to be greater than 1 due to the nature of the transformation.

The other advantage of the transformation over the conventional method is how certain real life processes rely on the win over loss value, rather than the original probabilistic value. For example, in chemical reactions, reaction rates are usually proportional to the concentration of the reactant divided by concentration of product

$$\frac{\mathrm{d}[\text{product}]}{\mathrm{d}t} \propto \frac{[\text{reactant}]}{[\text{product}]}. \tag{24}$$

Indeed, the paradox has already been considered in chemical reactions due to this relationship [19]. Such processes provide further justification for the use of our transformation.

5. Forming the new matrix

It may seem as though the original method requires only the extraction of certain entries and applying the formula in (12), while in our transformation we need to obtain an entire matrix before we proceed.

While we do require a transformation of the matrix, this transformation need not be fully completed for computational purposes. Suppose we have the stochastic matrix in (13). We can see that if we have the values $\{p_0, p_1, p_2\}$, then we may easily generate the matrix. Hence, in this section we will only deal with these three parameters. The purpose for this would be to consider for its numerical implementation, where we want to maximize its efficiency as far as possible.

Similarly, after the transformation, we obtain the parameters $\{p'_0, p'_1, p'_2\}$, where $p'_n, n \in \{0, 1, 2\}$ denotes the transformed values of probability. Again, it is easy to see that with these three parameters, we are able to generate the entire transformed stochastic matrix. In other words, our computation only revolves around these three parameters. For the purpose of numerical implementation, all we need are these three parameters, regardless of whether it is the original matrix, or the transformed one.

The next issue would be to determine the values of the parameters. In reality, we do not start with a stochastic matrix but a situation with initial conditions. Let us return to the simple example in the previous section. From this example, it is apparent that the computation of parameters for the original matrix is in no way simpler than computing for the transformed matrix. Therefore, forming this matrix, as well as using it for computational purposes, has no drawback compared to the original matrix. We will discuss the other advantages in the next section.

6. Introduction of bias factors

We will now introduce the bias factors $\epsilon_0, \epsilon_1, \epsilon_2 > 0$ in the form of a multiplier.

$$\mathbf{P} = \begin{pmatrix} \dfrac{\epsilon_2 p_2}{1-p_2} & \dfrac{1-p_2}{\epsilon_2 p_2} & 0 \\ 0 & \dfrac{\epsilon_0 p_0}{1-p_0} & \dfrac{1-p_0}{\epsilon_0 p_0} \\ \dfrac{1-p_1}{\epsilon_1 p_1} & 0 & \dfrac{\epsilon_1 p_1}{1-p_1} \end{pmatrix}. \tag{25}$$

It is clear whether these bias factors make a process win or lose.

Lemma 2. *Given Eq. (23) for a fair process, then*

$$\prod_{n=0}^{2} \epsilon_n \tag{26}$$

is greater than one if the process is winning, smaller than one if it is losing, and equal to one if it remains fair.

Proof. From Lemma 1, we know that for a fair process

$$\prod_{n=0}^{2} \frac{p_n}{1-p_n} = 1, \tag{27}$$

and since the bias factors are directly multiplied to the diagonal entries, we have

$$\left(\prod_{n=0}^{2} \frac{p_n}{1-p_n}\right)\left(\prod_{n=0}^{2} \epsilon_n\right) = \prod_{n=0}^{2} \epsilon_n, \tag{28}$$

and thus the product of the bias factors determines the overall product of the diagonal entries. In the conventional method, a modification of the original matrix with the bias factors entails a substitution of the new conditions into (12) each time this step is carried out. Our proposed method requires the initial product of the diagonals to be captured only once, whereas the conventional method requires all the entries of the original matrix to be stored for future use. Storing these entries and explicitly using them in computations is inefficient from the point of view of the memory and computational resources. Clearly, our proposed method is able to circumvent this inefficiency. □

We will now state the other advantages of these new bias factors.

Proposition 1. *The diagonal entries of the transformed stochastic matrix in (25), $p_{ii} \in \mathbb{R}_{\geq 0}$, $i \in \{0, 1, 2\}$ can be modified by bias factors to become any possible value $p'_{ii} \in \mathbb{R}_0^+$, $i \in \{0, 1, 2\}$ depicting the transformed probability of an outcome.*

Proof. Every term $\epsilon_0, \epsilon_1, \epsilon_2 > 0$ can be varied without restriction as long as it remains non-negative. This means that every non-zero term in the modified matrix in (23) will have a range from zero to positive infinity.

$$\frac{p_n}{1-p_n} \in \mathbb{R}_0^+. \tag{29}$$

We first note that if for two variables x and y

$$0 \leq x \leq y \leq 1. \tag{30}$$

Then we have

$$\frac{x}{1-x} \leq \frac{y}{1-y}, \tag{31}$$

and therefore we can perform an inverse transformation f^{-1} on (29) to give

$$f^{-1}(0) \leq f^{-1}\left(\frac{p_n}{1-p_n}\right) \leq f^{-1}(+\infty), \tag{32}$$

or equivalently,

$$0 \leq p_n \leq 1 \tag{33}$$

which encompasses every possible value to depict the probability of an event in the original definition. Hence the bias factors can modify a process to become any other process with stochastic matrices of the form in (23). □

Theorem 1. *Suppose we have process A', from a fair process A and bias factors $\{\epsilon_0, \epsilon_1, \epsilon_2\}$. It is possible to modify another fair process B into A, such that the new fair process can be modified by $\{\epsilon_0, \epsilon_1, \epsilon_2\}$ to become A'.*

Proof. The equation

$$\left(\prod_{n=0}^{2} \frac{p_n}{1-p_n}\right) = 1 \tag{34}$$

represents the condition for a process to be fair. Let p_n be the parameters for process B. We then transform it to another fair process A. Introducing bias factors $\{\epsilon'_0, \epsilon'_1, \epsilon'_2\}$, we require

$$\prod_{n=0}^{2} \frac{\epsilon'_n p_n}{1-p_n} = 1, \tag{35}$$

or equivalently,

$$\left(\prod_{n=0}^{2} \epsilon'_n\right)\left(\prod_{n=0}^{2} \frac{p_n}{1-p_n}\right) = 1 \tag{36}$$

to be true, which is satisfied if

$$\prod_{n=0}^{2} \epsilon'_n = 1. \tag{37}$$

From Proposition 1, we know that a process can be modified by bias factors to become another process. This represents the condition for those bias factors.

Next we have the parameters of process A' to be

$$\frac{\epsilon_n \epsilon'_n p_n}{1-p_n}, \tag{38}$$

which is the introduction of ϵ_n into process A. □

This means that it is possible for every configuration to start with the same fair process, and in turn allows for a more efficient comparison. Another major advantage of the transformation is that, from the bias factors transforming a fair process to another, it is possible to see that the bias factors indeed have not made the fair process win or lose. In the conventional Parrondo configuration, if we set $\epsilon = 0$, we have

$$\mathbf{P}_A = \begin{pmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{pmatrix} \tag{39}$$

$$\mathbf{P}_B = \begin{pmatrix} 0 & 0.1 & 0.9 \\ 0.25 & 0 & 0.75 \\ 0.75 & 0.25 & 0 \end{pmatrix}. \tag{40}$$

It is not obvious if B represents a fair process. Suppose we wish to compare the two processes A and B instead of mixing them. We can introduce $\epsilon = 0.005$ to have

$$\mathbf{P}_A = \begin{pmatrix} 0 & 0.495 & 0.505 \\ 0.505 & 0 & 0.495 \\ 0.495 & 0.505 & 0 \end{pmatrix} \quad (41)$$

$$\mathbf{P}_B = \begin{pmatrix} 0 & 0.095 & 0.905 \\ 0.255 & 0 & 0.745 \\ 0.745 & 0.255 & 0 \end{pmatrix}. \quad (42)$$

Furthermore, there is no rule which allows one process to be related to the other in a direct manner. Without a relation, any comparison will not be easy. However, our proposed transformation is able to provide the relationship between the two processes, so that comparison becomes direct.

As an illustration, we will refer to the conventional Parrondo configuration, in which the transformed stochastic matrices for $\epsilon = 0$ is given by

$$\mathbf{P}_A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \quad (43)$$

$$\mathbf{P}_B = \begin{pmatrix} 3 & \frac{1}{3} & 0 \\ 0 & \frac{1}{9} & 9 \\ \frac{1}{3} & 0 & 3 \end{pmatrix}. \quad (44)$$

It can be seen that by introducing bias factors into $\mathbf{P}_A$ in which

$$\{\epsilon_0, \epsilon_1, \epsilon_2\} = \{1/9, 3, 3\}, \quad (45)$$

we have converted $\mathbf{P}_A$ into $\mathbf{P}_B$. A simple multiplication of the bias factors shows that it is 1, and thus the modified process is still fair. Alternatively, by Lemma 1, it can be inferred that both processes are fair by taking the product of the diagonals as well. This also means that it is possible to start every process from (42), such as when we introduce bias factors into (42) with the values

$$\{\epsilon'_0, \epsilon'_1, \epsilon'_2\} = \{1/2, 2, 2\}, \quad (46)$$

then it is possible for the same matrix to be obtained from (41) by using (44)

$$\{\epsilon_0\epsilon'_0, \epsilon_1\epsilon'_1, \epsilon_2\epsilon'_2\} = \{1/18, 6, 6\}, \quad (47)$$

such that both matrices become

$$\mathbf{P}_A = \mathbf{P}_B = \begin{pmatrix} \frac{1}{18} & 18 & 0 \\ 0 & 6 & \frac{1}{6} \\ \frac{1}{6} & 0 & 6 \end{pmatrix}. \quad (48)$$

Corollary 1. *Given two processes A and B. If process B is modified by bias factors to become another process C, it is possible to relate processes A and C by combining the bias factors modifying from A to B with the bias factors modifying from B to C.*

It is therefore possible, through the transformation of the stochastic matrices, to relate two different processes by a set of bias factors. Not only does this allow for two different fair processes to be compared, it also shows the relationship between the bias factors.

7. Actual expected change in capital

We have presented the method to determine whether a process is winning or losing, and for the sake of completeness, we will proceed to determine the degree of its win or loss. It is generally not true that the greater the product of the diagonal of the transformed stochastic matrix as stated in (23), the greater the expected change in capital. We first obtain the expression for the expected change in capital per run in terms of the transformed probabilistic values. The most direct method would be to first perform an inverse transformation first, and then substitute into (7). That is, for a transformed stochastic matrix, where instead of (23) we represent it as

$$\mathbf{P} = \begin{pmatrix} p_2' & \frac{1}{p_2'} & 0 \\ 0 & p_0' & \frac{1}{p_0'} \\ \frac{1}{p_1'} & 0 & p_1' \end{pmatrix}. \tag{49}$$

We have

$$f^{-1}(p_n') = \frac{p_n'}{1+p_n'} \tag{50}$$

for $n = 0, 1, 2$. Substituting $f^{-1}(p_n') = p_n$ into (7), we find that

$$E(X) = \frac{3p_0'p_1'p_2' - 3}{3p_0'p_1'p_2' + 2(p_0'p_1' + p_0'p_2' + p_1'p_2') + 2(p_0' + p_1' + p_2') + 3}, \tag{51}$$

where X is a random variable such that it is one when heads is flipped and zero otherwise.

Now, we let

$$\kappa = 3p_0'p_1'p_2' + 2(p_0'p_1' + p_0'p_2' + p_1'p_2') + 2(p_0' + p_1' + p_2') + 3, \tag{52}$$

and will explore the properties of this newly defined expression.

Lemma 3. *For a fair process, the smallest possible value of* κ *is* 18.

Proof. We first recall that for a fair process $p_0'p_1'p_2' = 1$. Then, we have by the inequality of arithmetic and geometric means (AM–GM)

$$3p_0'p_1'p_2' + 2(p_0'p_1' + p_0'p_2' + p_1'p_2') + 2(p_0' + p_1' + p_2') + 3 > 3 + 6\sqrt[3]{\frac{1}{p_0'p_1'p_2'}} + 6\sqrt[3]{p_0'p_1'p_2'} + 3 = 18, \tag{53}$$

as was to be shown. □

Lemma 4. *If given a fixed value* $k > 0$ *such that*

$$\prod_{n=0}^{2} p_n' = k, \tag{54}$$

then the highest expected change in capital is produced when $p_0' = p_1' = p_2' = \sqrt[3]{k}$.

Proof. Equality of the AM–GM holds when every term is equal, that is,

$$\frac{p_0' + p_1' + p_2'}{3} \geq \sqrt[3]{p_0'p_1'p_2'}, \tag{55}$$

and similarly,

$$\frac{p_0'p_1' + p_1'p_2' + p_0'p_2'}{3} \geq \sqrt[3]{(p_0'p_1'p_2')^2}, \tag{56}$$

with equality when $p_0' = p_1' = p_2' = \sqrt[3]{k}$ for both (53) and (54). This would minimize κ, and thus maximize expected change in capital. □

In general, the greater the range of the values (the difference between the largest and the smallest value along the diagonal), the greater the value of κ. This intuition comes from the AM–GM inequality, because if two sets of numbers have the same geometric mean, then the set of numbers with the larger range would generally have a greater arithmetic mean. We will give a numerical example to illustrate this point. Consider the following transformed stochastic matrices $\mathbf{P}_1$ and $\mathbf{P}_2$,

$$\mathbf{P}_1 = \begin{pmatrix} 2 & 0.5 & 0 \\ 0 & 2 & 0.5 \\ 0.5 & 0 & 2 \end{pmatrix} \tag{57}$$

$$\mathbf{P}_2 = \begin{pmatrix} 4 & 0.25 & 0 \\ 0 & 2 & 0.5 \\ 1 & 0 & 1 \end{pmatrix}. \tag{58}$$

The product of the diagonals are the same ($2^3 = 4 \times 2 \times 1 = 8$), but the second matrix has a larger range ($2 - 2 < 4 - 1$). The expected change in capital for both matrices are found to be

$$E(X_1) = \frac{1}{3} \tag{59}$$

$$E(X_2) = \frac{21}{67}. \tag{60}$$

As expected, the former process produces a higher expected change in capital.

Theorem 2. *Given κ and the product of the diagonal entries in a modified stochastic matrix, the expected change in capital is a constant.*

Proof. The result follows from (49) and (50). Since κ and the product of the diagonal entries are fixed, the expected change in capital which is an expression in terms of these two values,

$$E(X) = \frac{3p'_0p'_1p'_2 - 3}{\kappa}, \tag{61}$$

will have a constant value. □

8. Random mix of processes in Parrondo's paradox

Apart from analyzing the processes individually, it is possible to consider a random mix of two different processes, and express it as a single stochastic matrix to be analyzed.

Proposition 2. *Given two modified stochastic matrices* **P** *and* **Q**,

$$\mathbf{P} = \begin{pmatrix} p'_2 & \frac{1}{p'_2} & 0 \\ 0 & p'_0 & \frac{1}{p'_1} \\ \frac{1}{p'_1} & 0 & p'_1 \end{pmatrix} \tag{62}$$

$$\mathbf{Q} = \begin{pmatrix} q'_2 & \frac{1}{q'_2} & 0 \\ 0 & q'_0 & \frac{1}{q'_1} \\ \frac{1}{q'_1} & 0 & q'_1 \end{pmatrix}, \tag{63}$$

the final stochastic matrix obtained from randomly mixing the two processes is

$$\mathbf{P} \oplus \mathbf{Q} = \begin{pmatrix} \frac{2p'_2q'_2 + p'_2 + q'_2}{p'_2 + q'_2 + 2} & \frac{p'_2 + q'_2 + 2}{2p'_2q'_2 + p'_2 + q'_2} & 0 \\ 0 & \frac{2p'_0q'_0 + p'_0 + q'_0}{p'_0 + q'_0 + 2} & \frac{p'_0 + q'_0 + 2}{2p'_0q'_0 + p'_0 + q'_0} \\ \frac{p'_1 + q'_1 + 2}{2p'_1q'_1 + p'_1 + q'_1} & 0 & \frac{2p'_1q'_1 + p'_1 + q'_1}{p'_1 + q'_1 + 2} \end{pmatrix}. \tag{64}$$

Proof. The original stochastic matrix representing the random mix of two processes $\mathbf{P}_C$ is the arithmetic mean of the two original stochastic matrices of the two processes. Hence if we have (8) and (9), then

$$\mathbf{P}_C = \frac{\mathbf{P}_A + \mathbf{P}_B}{2}. \tag{65}$$

Therefore if we perform an inverse transformation $f^{-1}(p'_n) = p_n$ and substitute into (63), we obtain (62). □

With this Proposition 2, we have shown that we can still use the transformed matrices to mix the two processes in a random order. This also allows us to prove Parrondo's paradox for the coin-flipping example.

For example, if we set $\epsilon = 0.05$ then the transformed matrices for **A** and **B** would be

$$\mathbf{P}_A = \begin{pmatrix} 0.9802 & 1.0202 & 0 \\ 0 & 0.9802 & 1.0202 \\ 1.0202 & 0 & 0.9802 \end{pmatrix} \quad (66)$$

$$\mathbf{P}_B = \begin{pmatrix} 2.9216 & 0.3423 & 0 \\ 0 & 0.1050 & 9.5263 \\ 0.3423 & 0 & 2.9216 \end{pmatrix}. \quad (67)$$

In a random mix C, we have the new stochastic matrix from Proposition 2,

$$\mathbf{P}_A \oplus \mathbf{P}_B = \begin{pmatrix} 0.4184 & 2.3898 & 0 \\ 0 & 1.6316 & 0.6129 \\ 0.6129 & 0 & 1.6316 \end{pmatrix}. \quad (68)$$

It can be easily verified that only the product of the diagonals for the stochastic matrix formed from the random mix is greater than one. We now proceed to the general proof. If we let the transformed diagonal entries in $\mathbf{P}_A$ be a_i and $\mathbf{P}_B$ be b_i, then as long as the following three conditions are satisfied:

$$\prod_{n=0}^{2} a_n < 1 \quad (69)$$

$$\prod_{n=0}^{2} b_n < 1 \quad (70)$$

$$\prod_{n=0}^{2} \frac{2a_n b_n + a_n + b_n}{a_n + b_n + 2} > 1 \quad (71)$$

then Parrondo's paradox is achieved. Note that for process with more than three states, say m, then the only change that has to be made would be to change the lower and upper limits in Eqs. (67)–(69) to be integers from zero to m.

9. Numerical example

In this section, we consider the second fair process in the coin-flipping example, with the stochastic matrix as shown.

$$\mathbf{P}_B = \begin{pmatrix} 0 & 0.1 & 0.9 \\ 0.25 & 0 & 0.75 \\ 0.75 & 0.25 & 0 \end{pmatrix}. \quad (72)$$

By substituting the values into (12), we deduce that the process is fair. We then introduce a set of bias factors, which is clearly indicated below.

$$\mathbf{P}_B = \begin{pmatrix} 0 & 0.1 - 0.005 & 0.9 + 0.005 \\ 0.25 - 0.005 & 0 & 0.75 + 0.005 \\ 0.75 + 0.005 & 0.25 - 0.005 & 0 \end{pmatrix}. \quad (73)$$

We have decreased the probability of winning when the player's capital is a multiple of three, but increased the chances of winning when it is not a multiple of three. If we want to determine whether this modified process is winning or losing, we have to repeat the entire calculation in (12). However, suppose we transform the process, and find that it is fair as the diagonals multiply to one,

$$\mathbf{P}_B = \begin{pmatrix} 3 & \frac{1}{3} & 0 \\ 0 & \frac{1}{9} & 9 \\ \frac{1}{3} & 0 & 3 \end{pmatrix}. \quad (74)$$

Similarly, we introduce a set of bias factors.

$$\mathbf{P}_B = \begin{pmatrix} (1.2) \cdot 3 & \frac{1}{(1.2) \cdot 3} & 0 \\ 0 & (0.8) \cdot \frac{1}{9} & \frac{9}{(0.8)} \\ \frac{1}{(1.2) \cdot 3} & 0 & (1.2) \cdot 3 \end{pmatrix}. \quad (75)$$

We have again increased the probability of winning when the player's capital is a multiple of three, and decreased the chances of winning when it is not. However, we can see that it is still possible to determine whether the process is winning or losing simply by checking the product of the diagonals. In addition, since the product is one, we may just directly multiply the bias factors together instead. This is more efficient than using the expression in (12).

10. Future work

Our proposed transformation can potentially be extended to more than two outcomes by a more complex transformation. Modifications to the transformation may include adding the natural logarithm

$$\mathbf{P} = \begin{pmatrix} \ln \frac{p_2}{1-p_2} & \ln \frac{1-p_2}{p_2} & 0 \\ 0 & \ln \frac{p_0}{1-p_0} & \ln \frac{1-p_0}{p_0} \\ \ln \frac{1-p_1}{p_1} & 0 & \ln \frac{p_1}{1-p_1} \end{pmatrix}, \tag{76}$$

and therefore allowing the bias factors to be introduce as a separate matrix, such as

$$\epsilon = \begin{pmatrix} \ln \epsilon_2 & -\ln \epsilon_2 & 0 \\ 0 & \ln \epsilon_0 & -\ln \epsilon_0 \\ -\ln \epsilon_1 & 0 & \ln \epsilon_1 \end{pmatrix}. \tag{77}$$

Another example would be when we let:

$$p'_n = \ln \frac{p_n}{1-p_n}, \tag{78}$$

then our modified stochastic matrix in (22) would be

$$\mathbf{P} = \begin{pmatrix} e^{p'_2} & e^{-p'_2} & 0 \\ 0 & e^{p'_0} & e^{-p'_0} \\ e^{-p'_1} & 0 & e^{p'_1} \end{pmatrix}. \tag{79}$$

It may also be useful to express the entries in the form

$$e^{p'_n} = \cosh p'_n + \sinh p'_n \tag{80}$$

$$e^{-p'_n} = \cosh p'_n - \sinh p'_n. \tag{81}$$

This would allow further simplification if the two values are added up.

These additional modifications may be useful for fields utilizing these functions when dealing with probability.

While our analysis considers the original formulation of Parrondo's paradox, other variations of the paradox do exist. The idea of applying a transformation will still be relevant, although the transformation may not be the same as the one derived in this paper. Possible transformations may include a trigonometric function such as $y = \tan(1/2 \cdot \pi x)$ and this motivates future work.

11. Conclusion

We have proposed an original and novel method for transforming stochastic matrices using a mapping f such that whenever $p_1 + p_2 = 1$, we instead have $f(p_1)f(p_2) = 1$. This has several advantages over the existing method of analyzing stochastic matrices. For instance, we are able to tell if the process is winning or losing simply by checking the product of the diagonal entries of the transformed stochastic matrices. A process is found to be winning if it is larger than one, losing if it is smaller and fair if it is equal to one. With our proposed method, the expected change in capital can be derived easily in terms of the transformed entries in the matrix too.

The main advantages of our proposed transformation are (1) being able to quickly determine whether a process is winning or losing by simple multiplication; (2) allowing a more systematic introduction of bias into fair processes compared to conventional methods while retaining the use of simple matrix operations; (3) being able to relate to real life better with its win over loss concept and; (4) the reduction of computation time when determining whether a process is winning or losing.

Our proposed method of analysis may be applicable in Refs. [8,9], as well as for quantum Parrondo games in general. It may also be better to adopt our proposed method for particles which stochastically move under potential gradients. This is because potential fields have a tendency to diminish exponentially, hence a non-linear transformation may prove to be a better model. Note that the paradox was formulated as a result of research in the Brownian ratchet, where a thought experiment allowed the extraction of energy from the random motion of particles. The analogy in Ref. [18] describes the similarity in configurations between the ratchet and its game-theoretic model, such as the presence of a source of potential. In the ratchet, the source of potential comes from electrostatics and gravity, which is found to be analogous to the rules of the games in Parrondo's paradox. In this aspect, our proposed method may prove advantageous to the user as potential gradients tend to be multipliers (the gradient of a straight line in an x–y coordinate system, for example, is a multiplier to its current value of x).

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