Allison mixture and the two-envelope problem

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In the present study, we have investigated the Allison mixture, a variant of the Parrondo's games where random mixing of two random sequences creates autocorrelation. We have obtained the autocorrelation function and mutual entropy of two elements. Our analysis shows that the mutual information is nonzero even if two distributions have identical average values. We have also considered the two-envelope problem and solved for its exact probability distribution.

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I. INTRODUCTION

Parrondo's paradox states that playing two losing games in a random or periodic order can result in a winning outcome [1–11]. Parrondo's games are related to the Brownian ratchets [12–16] with applications in physics, biology, engineering, and financial risk. Applications have also been linked to quantum models [17,18], where the classical coin toss is replaced by the measurement of a qubit. One of the studies is by Meyer *et al.* [19], in which they show that Parrondo's paradox can be modeled by probabilistic lattice gas automata. Their work introduces a quantum analog of the ratcheting mechanism seen in the original Parrondo's game, with possible applications in quantum computing. Further investigations on quantum Parrondo's games [20–24] yielded other variants, in the process shedding light on the roles of entanglement and coherence on game outcomes. Almeida *et al.* have considered how two chaotic systems can give rise to order, in the form of quadratic maps. This is related to Parrondo's paradox, having a lose + lose = win situation [25]. Other phenomena like the presence of stable states within chaos [26,27] have been understood under the framework of Parrondo's games. The paradox has also been considered in reliability theory. Starting with subunits of a system being less reliable than the subunits of another system, Crescenzo has shown that randomly choosing units from these two systems can allow one to obtain a system that is more reliable than the initial two [28].

In evolutionary biology, Wolf *et al.* have found that the success of bacterial random phase variation can be understood as a variant of Parrondo's paradox [29], in which random alternations between losing strategies (in this case, cells that choose the wrong phase variation or sequence of variations) produce a winning strategy. Separately, the evolution of less accurate sensors has been modeled and explained in terms of the paradox [30]. In ecology, the periodic alternation of certain organisms between nomadic and colonial behaviors has also been suggested as a manifestation of the paradox [31]. Recently, there is an intriguing finding of Parrondo-like phenomena in a Bayesian approach to the modeling of the work by a jury [32]: The unanimous decision of its members has a low confidence. The developments of Parrondo's paradox have cut across many disciplines, with a wide range of possible applications.

Another interesting variant of Parrondo's paradox is the Allison mixture [33–36], where random mixing of two random sequences creates autocorrelation [36]. This is an area which has not been fully explored yet. The Allison mixture has some resemblance to the thermodynamic picture of the Feynman-Smoluchowski ratchet: At equilibrium, there is detailed balance, time reversibility, and no net displacement, whereas out of equilibrium detailed balance is broken, with time irreversibility leading to net displacement [36,37]. It is worth noting that the Allison mixture process mixes two random sequences in a time-irreversible way *only* when symmetry in the governing transition probabilities is broken. Readers can refer to Refs. [8] and [37] for more information on the Allison mixture. There have been discussions to link the Allison mixture to applications in encryption and optimization of file compression [33].

In the two-envelope problem, the symmetry is also preserved if the envelopes are closed but broken when the envelopes are opened and an observation is being made [37]. The two-envelope problem has been explored, showing interesting similarities with volatility pumping on the stock market, modeling statistical distribution of words in a human language, language of information theory, and quantum game setting [37]; for the Schroedinger version of the game, see Ref. [38].

Both the Allison mixture and two-envelope problem possess counterintuitive dynamics as a result from symmetry breaking in discrete time ratchet phenomenon. They are closely related [39] and have motivated the present study. In order to gain further insight into the Allison mixture, we will analyze the correlation information between two elements of the Allison mixture and then solve for the information entropy by following the methods in Refs. [40] and [41]. In a similar

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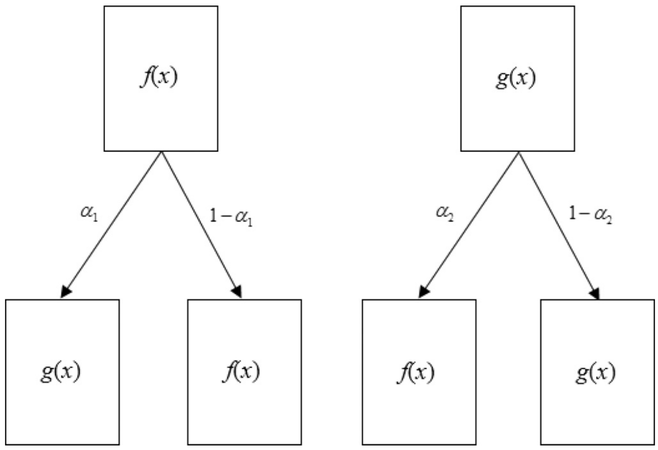

FIG. 1. The transition scheme for the Allison mixture. We show the processes and corresponding probabilities. $f(x)$, $g(x)$ are the distributions and α_1, α_2 are the transition probabilities.

vein, we will also solve for the exact probability distribution of the two-envelope problem.

II. THE ALLISON MIXTURE

In the Allison mixture, we have two random sequences with distributions $f(x)$ and $g(x)$ for the random numbers $x \equiv X(t)$ in the sequence for discrete time t. One of the sequences is chosen with a certain distribution. In each round, we either stick with the original choice with a probability $(1-\alpha_i)$ or move to the alternative distribution with a probability α_i. Figure 1 shows the scheme for the Allison mixture. The Markov chain governing the Allison mixture process has been well described in Ref. [33].

When the mean of the distributions are different, an autocorrelation arises:

$$\langle (X(t)-\langle X\rangle)(X(t+\tau)-\langle X\rangle)\rangle \neq 0 \tag{1}$$

for which

$$\alpha_1+\alpha_2 \neq 1, \alpha_1 \neq 0, \alpha_2 \neq 0. \tag{2}$$

Here, we calculate the entropy of the process. The transition probabilities will first be calculated. The transition matrix

$$\begin{pmatrix} 1-\alpha_1 & \alpha_2 \\ \alpha_1 & 1-\alpha_2 \end{pmatrix} \tag{3}$$

has two eigenvalues λ_1,λ_2,

$$\lambda_1 = 1, \quad \lambda_2 = 1-\alpha_1-\alpha_2 \equiv \lambda \tag{4}$$

with the corresponding eigenvectors:

$$\begin{gathered}(\alpha_2,\alpha_1),\\ (-1,1).\end{gathered} \tag{5}$$

We have the distribution in one position,

$$\rho_1(x) = \frac{\alpha_2 f(x)+\alpha_1 g(x)}{\alpha_1+\alpha_2}, \tag{6}$$

and transition probabilities $p_{ij}, i = 1,2$:

$$\begin{aligned} p_{11} &= \frac{\alpha_2+\alpha_1\lambda^t}{\alpha_2+\alpha_1}, \quad p_{12} = \frac{\alpha_1(1-\lambda^t)}{\alpha_2+\alpha_1},\\ p_{22} &= \frac{\alpha_1+\alpha_2\lambda^t}{\alpha_2+\alpha_1}, \quad p_{21} = \frac{\alpha_2(1-\lambda^t)}{\alpha_2+\alpha_1}. \end{aligned} \tag{7}$$

Now, consider the joint distribution of two values of the sequence, $\rho_2(x,y)$, where we denote $x \equiv X(t), y = X(t+\tau)$ and separate distributions $\rho_1(x),\rho_1(y)$,

$$\begin{aligned} \rho_2(x,y) &= \frac{\alpha_2 f(x)}{\alpha_1+\alpha_2}[p_{11}f(y)+p_{12}g(y)] + \frac{\alpha_1 g(x)}{\alpha_1+\alpha_2}[p_{21}f(y)+p_{22}g(y)]\\ &= \left[\frac{\alpha_2 f(x)}{\alpha_2+\alpha_1}+\frac{\alpha_1 g(x)}{\alpha_2+\alpha_1}\right]\left[\frac{\alpha_2 f(y)}{\alpha_2+\alpha_1}+\frac{\alpha_1 g(y)}{\alpha_2+\alpha_1}\right] + \frac{\lambda^t\alpha_2\alpha_1[f(x)f(y)+g(x)g(y)-f(x)g(y)-f(y)g(x)]}{(\alpha_2+\alpha_1)^2}\\ &= \rho_1(x)\rho_1(y)+\lambda^t\frac{\alpha_1\alpha_2}{(\alpha_1+\alpha_2)^2}Q(x,y),\\ Q(x,y) &= f(x)f(y)+g(x)g(y)-f(x)g(y)-g(x)f(y). \end{aligned} \tag{8}$$

From the latter expression, it is easy to derive for the autocorrelation:

$$\begin{gathered} \int \rho_2(x,y)x^l y^n dxdy - \int dxdy\rho_1(x)\rho_1(y)x^l y^n = (I_l I_n + J_l J_n - I_l J_n - I_n J_l)\frac{\alpha_1\alpha_2}{(\alpha_1+\alpha_2)^2}\lambda^t,\\ \int dx f(x)x^l = I_l, \int dx g(x)x^l = J_l. \end{gathered} \tag{9}$$

For $t = 1, l = 1, n = 1$, Eq. (9) coincides with the result of Ref. [33].

For illustrative purpose, we will construct $f(x)$ and $g(x)$ such that $I_1 = J_1$, $I_2 \neq J_2$. Let $f(x)$ be an uniform distribution in the interval $[0,1]$, and $g(x) = 2(1-2|x-0.5|)$ in the interval $[0,1]$. $f(x)$ and $g(x)$ will both have unit area in the domain $[0,1]$, and $I_1 = J_1 = 0.5$, but $I_1 = 1/3 \neq J_2 = 7/24$.

For $n = 2$, $l = 2$ and $\alpha_1 = \alpha_2 = 0.2$, we obtain in Fig. 2 an exponential decay of autocorrelation with respect to t for Eq. (9).

Consider the mutual entropy

$$I_m = -\int dxdy\rho_2(x,y)\ln\frac{\rho_2(x,y)}{\rho_2(x)\rho_1(y)} \tag{10}$$

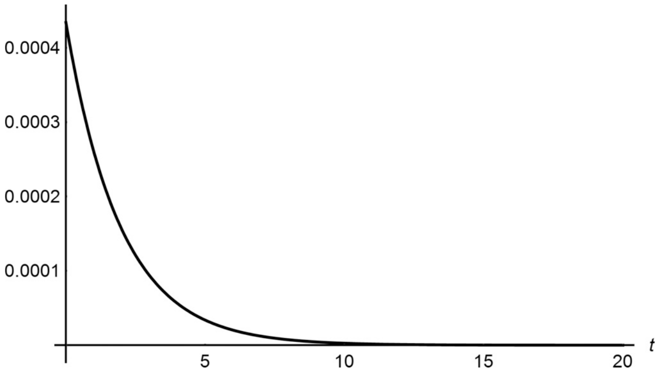

FIG. 2. Graph illustrating an exponential decay of autocorrelation with respect to t for Eq. (9) with $n = 2$, $l = 2$, and $\alpha_1 = \alpha_2 = 0.2$.

for which the weak correlation case can be approximated by

$$I_m = \left\langle \frac{\rho_2(x,y)}{\rho_2(x)\rho_1(y)} - 1 \right\rangle = \frac{\lambda^{2t}(\alpha_1\alpha_2)^2}{(\alpha_1+\alpha_2)^4} \int dxdy \frac{Q^2(x,y)}{\rho_1(x)\rho_1(y)}. \quad (11)$$

For the simple case of $\alpha_1 = \alpha_2$, we will have

$$I_m = \lambda^{2t} \int dx \times \frac{[f(x)f(y) + g(x)g(y) - f(x)g(y) - f(y)g(x)]^2}{[f(x)+g(x)][f(y)+g(y)]}. \quad (12)$$

It is easy to see that the autocorrelation disappears at $I_1 = J_1$. The work in Ref. [33] assumed that the nonzero correlation arises when two distributions have different averages. We have now discovered that the mutual entropy depends on the difference of the distributions, even if the averages are identical; some correlation exists in the sequence. This fundamental understanding is likely to improve the applicability of the Allison mixture in modeling real life problems. We will now proceed to investigate the two-envelope problem.

III. TWO-ENVELOPE PROBLEM

Next, consider the following game. There are two envelopes. There is money x in one envelope and $2x$ in the other envelope, where x is obtained from some distribution, $\rho(x)$. In each round, the player randomly chooses an envelope and observes the amount inside. The question is, should the player keep that envelope or now swap it for the other one, in order to maximize payoff? Using the exact analogy given in Ref. [37], if the observed amount is $y \in (x,2x)$, then the other envelope must contain either $2y$ or $y/2$ dollars. One may be inclined to swap because this will mean either one gains by a net y dollars or drops back by $y/2$ dollars. The same argument holds if the other envelope was selected first [37]. Any apparent paradox is generally due to treating what is actually a conditional probability as an unconditional probability [42]. In this paper, we focus on the two-envelope problem attributed by the information theorist Cover [43]; for the Schredinger version of the model, see Ref. [38]. The amount x is unbounded and there is a large ensemble of independent games.

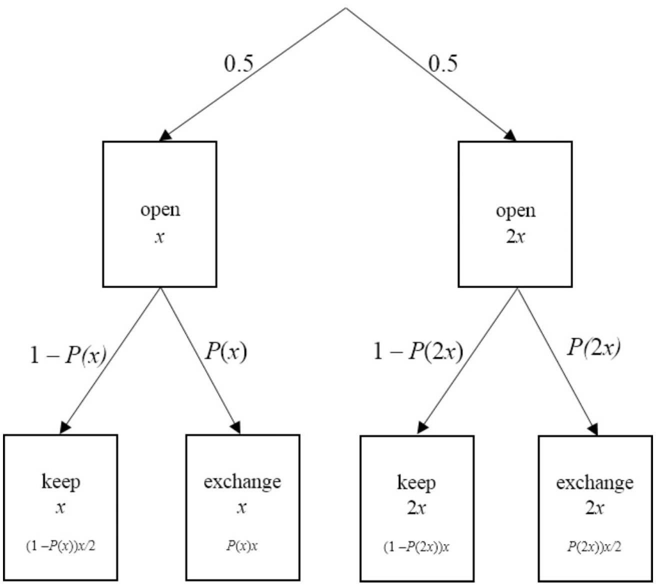

FIG. 3. The transition scheme for the two-envelope problem, related to Eq. (13). We show the processes and corresponding probabilities. In the lower boxes, we give the corresponding average change of capital according to Eq. (13).

The average rate of capital has been calculated in Ref. [37] and some mathematical results have been derived using functional equations as part of an optimization problem. Here, we will give an integral representation for the exact probability distribution of the model, then calculate both the mean capital growth rate and variance of the capital distribution after a large number of games.

We first assume that the player can either take the money from the envelope that has been opened or choose another envelope. The choice is described via some probability function, $P(x)$. We need to find the average rate and variance for the player's capital after t rounds, for t large. The rate calculated in Ref. [37] makes use of Cover's switching function [43]. It makes a biased random choice where the bias is conditioned on the observed value of one of the states.

Let us now consider an ensemble of distributions, with probability $p(t,x)$ of having capital x after t trials. When $\rho(x) = \delta(X - x)$,

$$p(t+1,z) = p(t,z-X)\frac{1-P(X)}{2} + p(t,z-2X)\frac{P(X)}{2} + p(t,z-2X)\frac{1-P(2X)}{2} + p(t,z-X)\frac{P(2X)}{2}. \quad (13)$$

We use the probabilities $\frac{1-P(X)}{2}, \frac{P(X)}{2}$ and $\frac{1-P(2X)}{2}, \frac{P(2X)}{2}$ as the player chooses the envelope randomly. Figure 3 illustrates the scheme for the two-envelope problem as described above, taking into account Eq. (13).

Similarly, we can obtain the general distribution in the same manner:

$$\begin{aligned} p(t+1,z) &= \int dx\rho[x]\left[p(t,z-x)\frac{1-P(x)}{2} + p(t,z-2x)\frac{P(x)}{2} + p(t,z-2x)\frac{1-P(2x)}{2} + p(t,z-x)\frac{P(2x)}{2}\right] \\ &= \int dx\rho[x]\left[p(t,z-x)\frac{1-P(x)}{2} + p(t,z-x)\frac{P(2x)}{2}\right]\int dx\rho[x/2]p(t,z-x)\frac{P(x/2)}{4} + p(t,z-x)\frac{1-P(x)}{4}. \end{aligned} \tag{14}$$

If we consider the process on the infinite axis (there is no restriction in money supply), we can always write a Fourier transformation [41]:

$$\begin{aligned} P(t,z) &= \int_{-\pi}^{\pi} dk e^{iknz}\bar{P}(k,t), \\ \bar{P}(k,t) &= \frac{1}{2\pi}\int dz P(t,z)e^{-ikz}. \end{aligned} \tag{15}$$

We will make use of the analogy between an abundance of money supply and particle to aid in our derivations here. Assume that the particle starts at $n=0$ and $\bar{P}(k,0)=\frac{1}{2\pi}$.

Equation (14) can be easily transformed into

$$\bar{P}(k,t+1) = e^{tK(-ik)}\bar{P}(k,t), \tag{16}$$

where we denote $p=-ik$,

$$\begin{aligned} e^{K(p)} &= \int dx\rho[x]e^{-px}\left[\frac{1-P(x)}{2} + \frac{P(2x)}{2}\right] \\ &\times \int dx\rho[x/2]e^{-px}\left[\frac{P(x/2)}{4} + \frac{1-P(x)}{4}\right]. \end{aligned} \tag{17}$$

We can then write the solution as

$$P(t,z) = \int_{-\infty}^{\infty} dk e^{tK(ik)+ikz}\bar{P}(k,0). \tag{18}$$

Thus, we have obtained an integral representation for the exact probability distribution after t rounds, akin to the solution to Parrondo's games [40]. It will now be useful to calculate the average rate of capital growth and the variance of capital distribution after t rounds to have a better understanding of the process.

Let us use an expansion

$$K(p) = -rp + vp^2/2, \tag{19}$$

and Eqs. (18) and (19) give

$$\begin{aligned} r &= \int dx\rho[x]x\left[\frac{1-P(x)}{2} + \frac{P(2x)}{2}\right] \\ &\times \int dx\rho[x/2]x\left[\frac{P(x/2)}{4} + \frac{1-P(x)}{4}\right]. \end{aligned} \tag{20}$$

Equation (15) says that r is the average growth rate of capital: After t rounds of the game, the player has a gain rt. In a similar way, we can also derive an expression for v:

$$\begin{aligned} (r^2+v) &= \int dx\rho[x]x^2\left[\frac{1-P(x)}{2} + \frac{P(2x)}{2}\right] \\ &\times \int dx\rho\left[\frac{x}{2}\right]x^2\left[\frac{P\left(\frac{x}{2}\right)}{4} + \frac{1-P(x)}{4}\right], \end{aligned} \tag{21}$$

where $V(t)\equiv vt$ is the variance of the capital distribution after t rounds.

For the simple model by Eq. (13), Eqs. (20) and (21) give

$$\begin{aligned} r &= [1.5-(p_2-p_1)/2]X, \\ v &= [0.25-(p_2-p_1)^2/4]X^2, \end{aligned} \tag{22}$$

where $p_1\equiv P(X), p_2 = P(2X)$.

Figure 4 depicts the graph for the mean capital variance $V(t)\equiv\langle z^2\rangle - \langle z\rangle^2$ with respect to time t for the model given by Eq. (14) at $X=1$, whereas the solid dots are given by Eq. (22) for $P(1)=0.2, P(2)=0.3$. They are clearly in agreement with each other.

It is not difficult to generalize the model. We can simply choose between two correlated random numbers. When we observe one of them, we naturally obtain information about the second one as well. In such a case, we can construct an algorithm to obtain a better choice among these random assets. We can then formulate the model for the general distribution $\rho(x_1,x_2)$ and minimize the variance.

IV. DISCUSSION

Gunn *et al.* have previously provided an initial information-theoretic analysis of the Allison mixture [44]. In the present study, we have developed their work by obtaining the correlation information between two elements of the Allison mixture. In Ref. [33], it is assumed that the nonzero correlation arises when two distributions have different averages; it also assumes that the Allison mixture can create the correlations as observed in the context of DNA. Our results have now shown that that the correlation entropy depends on the difference between the

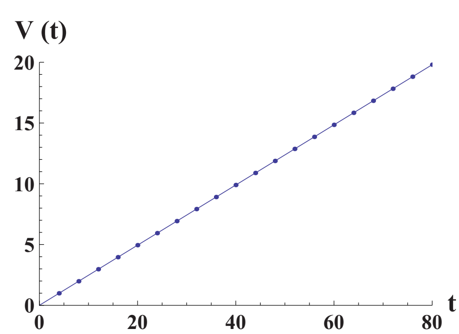

FIG. 4. The mean capital variance $V(t)\equiv\langle z^2\rangle - \langle z\rangle^2$ versus time t for the model by Eq. (14) with $X=1$. The smooth line is given by our numerics using Eq. (14), whereas the solid dots are due to analytics, Eq. (22). $P(1)=0.2, P(2)=0.3$.

distributions; some form of correlation exists in the sequence even if the averages are identical. As the correlation decreases exponentially, we cannot explicitly describe the observed scaling in the correlation of DNA sequence. It has been observed that the Markov-based models of the language have failed to create such long-range correlations [45].

In a similar vein, we have also considered the two-envelope problem and solve for its exact probability distribution. Probability distribution fitting related to the two-envelope problem can now be carried out for certain phenomenon. Predictive analysis can then take place, for instance, to forecast the frequency of occurrence of the magnitude of the phenomenon in a certain interval of interest. It can also be used to obtain the unknown parameters of the model describing certain data. Finally, we hypothesize that our analytical derivations in here can provide new insights on the conditions required to model a variety of real-life problems involving the Allison mixture or two-envelope problem more closely.

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