Supplementary Information

Cross-Issue Solidarity and Truth Convergence in Opinion Dynamics

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1 Analytical results

The information here is provided to orientate readers on existing analytical results, and how our own results build upon them. Here we restate some relevant results about the conditions required for consensus in previously developed models, followed by an analysis of the likelihood of consensus under the inclusivist and exclusivist models we have proposed.

Definition 1.1 (Consensus)
A group of agents with opinions $x(t)$ is said to reach consensus if

$$\lim_{t\to\infty} x(t) = c \tag{1}$$

where $c \in \mathbb{R}^m$ is some constant vector of opinions. More generally, we say that the agents reach consensus along dimension k if

$$\lim_{t\to\infty} x_k(t) = c_k \in \mathbb{R} \tag{2}$$

and that they reach multi-issue consensus if consensus is reached along every dimension k.

1.1 Conditions for consensus

In general, consensus is reached only if the graph of agents is "sufficiently connected". The following two results formalize that intuition. Detailed explanations and proofs are given in Hegselmann & Krause (2002)[1].

Theorem 1.1 (Consensus in the classical model)
A sufficient condition for consensus in the classical model is that the stochastic matrix A be irreducible. In other words, the directed graph of agents must be strongly connected. In this case, consensus will occur regardless of the initial opinion profile.

Theorem 1.2 (Consensus under bounded confidence in one dimension)
A necessary condition for consensus under bounded-confidence is that at every time t, there is always some *chain of confidence* connecting every pair of agents i and j. That is, a series of other agents $k_1, k_2, ..., k_r$ can be found between i and j such that $|x^i - x^{k_1}| < \epsilon$, $|x^j - x^{k_r}| < \epsilon$, and $|x^{k_q} - x^{k_{q+1}}| < \epsilon$ for all $1 \leq q \leq r$.

It is quite straightforward to extend this result to multiple dimensions by generalizing the chain of confidence to mean any series of agents of which consecutive pairs are in each others neighborhoods of acceptance. We thus have the following corollary:

Corollary 1.2.1 (Consensus under bounded confidence in multiple dimensions)
A necessary condition for consensus is that at every time t, for every pair of agents i and j, a series of other agents $k_1, k_2, ..., k_r$ can be found such that $i \in I(k_1, x)$, $j \in I(k_r, x)$, and $k_q \in I(k_{q+1}, x)$ for all $1 \leq q \leq r$.

A trivial consequence of these two results is that if there are only two interacting agents, they must both be within each others' neighborhoods of acceptance for convergence to be achieved. In fact, this is a sufficient condition for consensus, because no other agents can influence the pair such that they move away from each other.

1.2 Likelihood of consensus in relation to inclusivity

We can now state some results about the likelihood of consensus in our proposed multidimensional models. It has already been noted in the main text that exclusivism reduces the chances of consensus and inclusivism improves it. For the case of $n = 2$ agents, this can be formalized as follows:

Proposition 1.1
Suppose there are two agents with initial opinion uniformly randomly distributed across the (finite) opinion space. For any number of issues $m \geq 1$, increasing the number of issues to $m+1$ reduces the chances of the two agents reaching consensus if exclusivist interaction is employed. On the other hand, the chances of consensus are increased if inclusivist interaction is employed.

Proof. Without loss of generality, we assume an opinion space of $[0,1]^m$. In the exclusivist model, when m is increased to $m+1$, the neighborhood of acceptance around the first agent, $I_\cap(1,x)$, decreases in volume, because one has to take the intersection of $I_\cap(1,x)$ with $I_{m+1}(1,x)$ to create the new neighborhood of acceptance, and the intersection is smaller in volume. Furthermore, $|I_\cap(1,x)|$ is exactly equal to the probability that consensus will be reached, since consensus occurs if and only if the second agent falls within $I_\cap(1,x)$.

In the inclusivist model, the neighborhood of confidence $I_\cup(1,x)$ is increased when m increases to $m+1$, because one takes the union of $I_\cup(1,x)$ together with $I_{m+1}(1,x)$ to get the new neighborhood of acceptance, and the union is larger in volume. Thus, the likelihood of two-agent consensus increases. This completes the proof.

□

We can further generalize this result to encompass interactions with any degree of inclusivity α, assuming that α and the latitude of acceptance ϵ are constant across issues (i.e., $\alpha = \alpha_1 = ... = \alpha_m$ and $\epsilon = \epsilon_1 = ... = \epsilon_k$). First we state the relationship between α and the volume of the neighborhood of acceptance.

Theorem 1.3 (Volume of acceptance V_m as a function of α, ϵ, and the number of issues m)
Let ${}^mI^*(i,x)$ be the neighborhood of acceptance around agent i when there are m issues, and let V_m be the volume of this neighborhood (i.e. the amount of opinion space that falls within ${}_mI^*(i,x)$). Assuming that none of the neighborhood of acceptance falls outside of the opinion space, V_m can then be expressed as the recurrence relation

$$V_m = 2(\alpha-1)\epsilon V_{m-1} + \alpha^{m-1}(2\epsilon)^m \tag{3}$$

Solving this recurrence gives a closed-form expression for V_m

$$V_m = [\alpha^m - (\alpha-1)^m](2\epsilon)^m \tag{4}$$

and the ratio between V_{m+1} and V_m can simply be stated as

$$\frac{V_{m+1}}{V_m} = 2\epsilon\left(\alpha - 1 + \frac{\alpha^m}{\alpha^m - (\alpha-1)^m}\right) \tag{5}$$

Proof. For generalized inclusivity in m dimensions, the neighborhood of acceptance is defined as

$${}^mI^*_k(i,x) := \{j : |x^j_k - x^i_k| \leq \epsilon \text{ and } |x^j_l - x^i_l| \leq \alpha\epsilon \text{ for all } l \neq k\} \tag{6}$$

$$ {}^{m}I^*(i,x) := \bigcup_{k=1}^{m} {}^{m}I_k^*(i,x) \tag{7} $$

Notice that we can rewrite Equation 7 recursively as

$$ {}^{m}I^*(i,x) := {}^{m-1}I^*(i,x) \cup {}^{m}I_m^*(i,x) \tag{8} $$

By definition, ${}^{m-1}I^*(i,x)$ spans a volume of V_{m-1} in $(m-1)$ dimensional space. However, in m dimensional space, it ends up spanning a volume of $2\alpha\epsilon V_{m-1}$, because there is now one more dimension, namely m, where agents can fall within the expanded latitude of acceptance $\alpha\epsilon$ (i.e., whereas $|x_l^j - x_l^i| \leq \alpha\epsilon$ was only possible for all $l \neq k$, $1 \leq l \leq m-1$ in $(m-1)$ dimensions, in m dimensions, this becomes possible for all $l \neq k$, $1 \leq l \leq m$).

Next, notice that ${}^{m}I_m^*(i,x)$ spans a volume of $\alpha^{m-1}(2\epsilon)^m$ in m dimensional space. This is because any agent j can fall within ϵ of agent i for issue m and within $\alpha\epsilon$ for all other issues to be included within $I_m^*(i,x)$, giving a hyper-rectangle of length 2ϵ on one side and length $2\alpha\epsilon$ on $(m-1)$ sides. Note as well that that the intersection between ${}^{m}I_m^*(i,x)$ and ${}^{m-1}I^*(i,x)$ spans a volume of $2\epsilon V_{m-1}$, since the volume of $I_m^*(i,x)$ is of length 2ϵ in dimension m, and the cross-section of ${}^{m-1}I^*(i,x)$ in dimensions 1 to $(m-1)$ has a volume of V_{m-1}.

Combining these together, we have Equation 3:

$$ \begin{aligned} V_m &= 2\alpha\epsilon V_{m-1} - 2\epsilon V_{m-1} + \alpha^{m-1}(2\epsilon)^m \\ &= 2(\alpha-1)\epsilon V_{m-1} + \alpha^{m-1}(2\epsilon)^m \end{aligned} $$

It can easily be verified by induction that Equation 4 solves Equation 3, with the base case $V_1 = \epsilon$. Finally, Equation 5 is a direct consequence of Equation 4 and this completes the proof.

□

From these equations, we can determine the minimal degree of inclusivity α^* such that likelihood of consensus will always increase with the number of issues m.

Corollary 1.3.1 (Threshold of inclusivity α^*)
Suppose there are two agents with initial opinion distributed uniformly at random across the opinion space, and the neighborhood of acceptance of at least one agent is contained within the space. Then for any m, if the number of issues is increased to $m+1$, we can define a threshold of inclusivity

$$ \alpha^* = \frac{1}{2\epsilon} \tag{9} $$

above which the likelihood of consensus will also increase. Conversely, if $\alpha < \alpha^* = \frac{1}{2\epsilon}$, there exists some m above which the two-agent likelihood of consensus decreases when the number of issues is increased.

Proof. Assume without loss of generality an opinion space of $[0,1]^m$. Then, as noted before, the likelihood of consensus is exactly equal to V_m, because under the uniform distribution, V_m is the probability with which the second agent will fall within the first agent's neighborhood of acceptance.

The result follows from Equation 5, because if $V_{m+1}/V_m > 1$, then the likelihood of consensus will increase with the number of dimensions. Firstly, note that $\frac{\alpha^m}{\alpha^m - (\alpha-1)^m} \geq 1$, and so we have

$$ \frac{V_{m+1}}{V_m} \geq 2\epsilon\alpha $$

as a lower bound for any m. Secondly, as $m \to \infty$, we have $\frac{\alpha^m}{\alpha^m - (\alpha-1)^m} \to 1$, giving us

$$ \lim_{m\to\infty} \frac{V_{m+1}}{V_m} = 2\epsilon\alpha $$

It follows that $V_{m+1}/V_m \geq 1$ if $\alpha \geq \frac{1}{2\epsilon}$, but that V_{m+1}/V_m will eventually drop below 1 if $\alpha < \frac{1}{2\epsilon}$. This completes the proof.

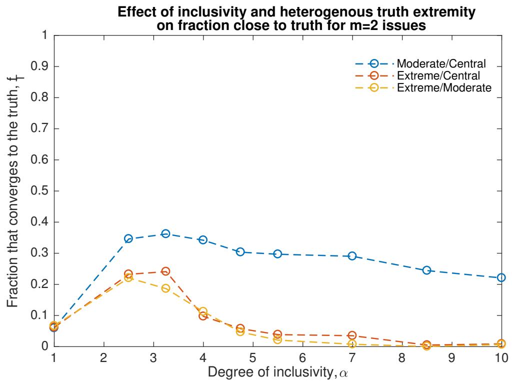
□

In other words, for a given latitude of acceptance (or level of open-mindedness) ϵ, there is a fixed degree of inclusivity α^* that will ensure consensus never becomes less likely as more issues get discussed. In fact, above this degree of inclusivity, consensus can become more likely, because participants discover more potential areas of agreement as they explore more issues, which then promotes engagement on other issues that they differ on.

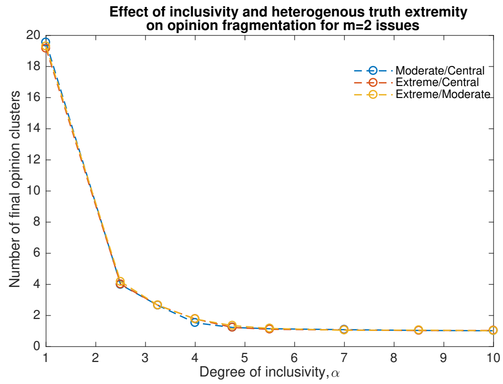
As might be expected, α^* varies inversely with ϵ. With low $\epsilon = 0.1$, α needs to be at least $\alpha^* = 5$ for the likelihood of consensus to be maintained — i.e., participants need to increase their willingness to engage by five-fold upon finding common ground, to compensate for the difficulty of finding common ground in the first place. With high $\epsilon = 0.5$, no further increase in willingness is necessary for consensus to remain likely ($\alpha^* = 1$), because participants are already open-minded to begin with.

2 Heterogenous truth extremity

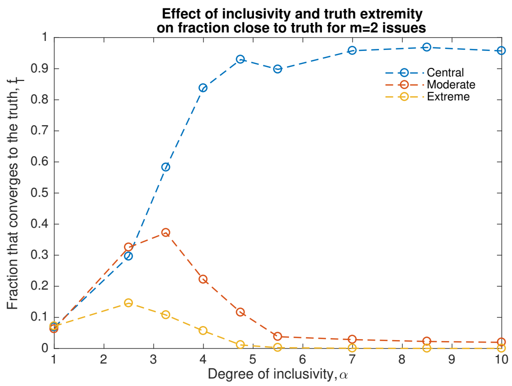
As mentioned in the main paper, the trend of truth convergence decreasing with truth extremity holds even when truth extremity is heterogeneous – i.e. when the truth is more extreme along one dimension than another. This is shown in Figure S1.



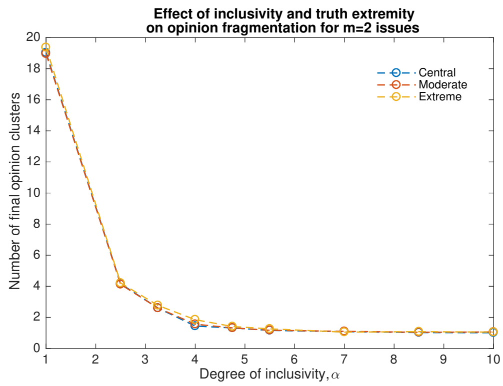
(a) f_T for heterogeneous extremity



(b) n_c for heterogeneous extremity



(c) f_T for homogeneous extremity



(d) n_c for homogeneous extremity

Figure S1: Effects of inclusivity and truth extremity on the fraction that converges to the truth f_T and the number of clusters n_c after $T = 100$ timesteps. Other parameters were $n = 50$, $\epsilon_k = 0.1$ for all k. Results were averaged over 100 trials. For the heterogeneous cases, 'Moderate/Central', 'Extreme/Central', and 'Extreme/Moderate' correspond to the truth being located at $(0.30, 0.45)$, $(0.15, 0.45)$, and $(0.15, 0.30)$ respectively.

As can be seen in Figure S1a, when the truth is moderate along one dimension and central along another (blue line), convergence towards the truth is lower than when the truth is central along both dimensions (compare with blue line in S1c). On the other hand, there is more convergence than when the truth is moderate along both dimensions (red line in S1c). This, in turn, has more convergence when the truth is extreme along one dimension and moderate along the other dimension (yellow line in S1a).

In general, as the distance of the truth from the center increases, convergence towards the truth decreases. When the truth is sufficiently extreme, increasing the degree of inclusivity α results in lower convergence (as in the case of extreme/central truth and extreme/moderate truth). However, as long as the truth is sufficiently central, a significant amount of convergence is maintained even as α increases. Finally, as can be seen by comparing Figures S1b and S1d, heterogeneous truth extremity has a negligible effect on the degree of opinion fragmentation, in accordance with the observation that truth extremity in general has little such effect.

3 Supplementary discussion

For readers familiar with the literature on opinion dynamics and political disagreement, we summarize some similarities and differences our present study has with related work, in order to give credit where due, and to better contextualize the novelty of our own contributions.

Firstly, though not explicitly focused on the study of cross-issue interactions, Baldassarri and Bearman's work on political polarization provides perhaps the richest model of multi-issue opinion dynamics thus far[2]. In their model, pairs of agents interact with each other if they are both highly interested in a specific issue. If their views on this issue are initially in opposition, then their opinions will only move closer if they already agree on a majority of other issues (i.e., compromise). Otherwise, if they disagree on a majority of other issues, their opinions on the discussed issue will move further apart (i.e., conflict). Through these mechanisms, their model successfully explained the observation that perceived levels of polarization often do not match the actual levels of difference. However, the study did not directly investigate the effect of the number of issues upon fragmentation or consensus, nor did it examine the conditions that under which cross-issue solidarity might be maintained.

Secondly, we note that the exclusivist model of cross-issue interaction is equivalent to previous multi-dimensional models of bounded confidence which use the ∞-norm as a measure of opinion distance[3]. Our contribution lies in situating this model in a more general framework of cross-issue interaction, thereby providing a new interpretation and expanding its applicability.

Lastly, we note the similarities between our model of generalized inclusivity and the two-issue bounded confidence model proposed by Huet et al.[4], where agreement on one issue expands the range of tolerance for disagreement on another. However, their model focuses on rejection effects, based on the idea that sufficiently strong disagreement on one issue can lead to opinions moving apart on the other. Our general model instead addresses a gap in understanding how agreement could promote more agreement — that agreeing on one issue might make individuals not just more *tolerant* of differences on other issues, but also more *accepting* and willing to change their minds.

References

[1] Hegselmann R, Krause U *et al.* 2002 *Journal of Artificial Societies and Social Simulation* **5**

[2] Baldassarri D and Bearman P 2007 *American sociological review* **72** 784–811

[3] Lorenz J 2008 Fostering consensus in multidimensional continuous opinion dynamics under bounded confidence *Managing complexity: insights, concepts, applications* (Springer) pp 321–334

[4] Huet S, Deffuant G and Jager W 2008 *Advances in Complex Systems* **11** 529–549