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REVIEW

Parrondo’s paradox from classical to quantum: A review

Joel Weijia Lai · Kang Hao Cheong

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Abstract Two losing games can be combined in a certain manner to give a winning outcome—this is known as Parrondo’s paradox. Parrondo’s paradox has found its applications across different disciplines such as physics, biology and finance, amongst others. At the turn of the millennium, there has been immense attention on the quantum Parrondo’s games as classical games are simulated using quantum notation. This review paper traces the construction of quantum Parrondo’s games from classical capital-dependent, history-dependent and cooperative Parrondo’s games. Directions for future research will also be discussed.

Keywords Quantum games · Parrondo’s paradox · Quantum decoherence

1 Introduction

J.M.R. Parrondo conceptualized the idea that two losing games can combine to give a winning game. This seemingly paradoxical result is called Parrondo’s paradox [1]. Parrondo’s initial motivation is to explain the dynamics of the Brownian ratchet [2,3]—a thought experiment about a mechanical machine driven by the Brownian motion of air molecules that can purportedly extract energy from random heat motions popularized by physicist Richard Feynman [4].

Viewed through a biological lens, the Parrondo’s paradox has been used to explain the underlying dynamics behind many biological phenomena [5–10]. Beyond biological systems, Parrondo’s paradox has also been developed in the field of economics (game theory) [11–17], computer science [18,19], physical chemistry, [20], chaos and nonlinear dynamics [21,22] and engineering [23–26]. Besides its theoretical advancements [27–31], specific problems in social dynamics [32–34], population dynamics [35], evolutionary biology [36] and ecology [37,38] have been modelled and explained in terms of the paradox.

However, at the core of these applications still lies the traditional game designs which can be classified into three main types: Capital-dependent, History-dependent and Cooperative Parrondo’s games. Each of these has been well studied over the past decades and are summarized, with its applications in the following review papers [5,39–41]. Some of the main derivations can be found in Ref. [42].

In this review paper, we will classify the traditional Parrondo’s games and its applications as “classical” as they fall outside the domain of quantum computing and quantum information. Consequently, any research with quantum information theory at its core motivation will be considered “quantum”, with an emphasis on the quantum Parrondo’s paradox.

J. W. Lai · K. H. Cheong (✉)
Science and Math Cluster, Singapore University of Technology and Design, 8 Somapah Road, Singapore 487372, Singapore
e-mail: kanghao_cheong@sutd.edu.sg

Another perspective of the Parrondo's paradox is to consider a losing game that becomes winning as a result of some form of "noise". This "noise" is another losing game. The recent rise in quantum game theory and the representation of biased coins as qubits have led to the development of quantum Parrondo's games. In quantum games, "noise" is inherently built into the system with the probabilistic nature of superposition of qubit states in quantum mechanics. The chaotic and nonlinear phenomena derived in classical systems are found in classical Parrondo's games. Therefore, the development of quantum Parrondo's games from classical games can open an avenue for the development of quantum chaotic games and nonlinear dynamical systems.

1.1 Classical theory to quantum theory

To understand the development of quantum Parrondo's paradox from classical, we first need to understand the link between classical and quantum theory—this is schematically summarized in Fig. 1.

A quantum theory is defined as a pair of Hilbert space $\mathcal{H}$ and Hamiltonian $\hat{H}$, which is a self-adjoint time operator in $\mathcal{H}$, generator of time evolution. Similarly, a classical theory is defined as a pair containing a phase space $\mathcal{M}$ and a classical Hamiltonian H. What we typically measure is an effective classical system that models an observation made on a quantum system. The process of obtaining an effective classical theory from an observation in quantum theory is called the "classical limit". The classical limit is the effective result of two factors, *decoherence* and *noise*, which are attributed to imperfection in measurements. The process of inverting a classical theory by mapping it to a quantum theory is call "quantization". Effectively, quantization is inverting a classical limit. However, this is not a well-posed problem because there exists multiple pairs of quantum theories that can give the same classical limit.

Our universe is inherently quantum mechanical. However, we perceive the universe in its "classical limit", described by classical theories. It is widely accepted that the developments in quantum mechanics have opened possibilities in many areas of applied physics and engineering that far supersede that of classical mechanics. Therefore, if we take the same approach to develop quantum Parrondo's games from classical Parrondo's games, we may be able to achieve more than what has been discovered so far. Intuitively, the first step is to model classical games using quantum mechanics language in the form of commutative algebra and to recreate classical results in the classical limit. This is the motivation for this review paper as we explore how quantum Parrondo's games are constructed from classical Parrondo's games. This is then extended to include aspects of chaotic and dynamical systems. The increasing applicability, availability and reliability of quantum computers and the motivation to achieve quantum supremacy [44–46] can open up possibilities of implementing calculations on quantum computers. Of particular interest are quantum game theory and quantum Parrondo's games—with application ranging from quantum supremacy to a wider class of computational problems in chaotic and dynamical systems of higher order complexities.

2 Constructing quantum from classical Parrondo's games

The increasing interest in quantum information and field theoretic methods to model classical problems and the development of quantum game theory has opened up the field of quantum Parrondo's paradox [47]. Quantum theoretic methods are advantageous over classical methods in certain computations [48], thus it is advantageous to explore the possibilities of employing quantum computing methods to develop existing fields of classical Parrondo's games and apply it to systems that may possibly be limited by classical computation. In the following sections, we will review the developments in translating classical Parrondo's games into quantum Parrondo's games.

2.1 Capital-dependent Parrondo's games

Capital-dependent games, or position-dependent games, are the first seminal Parrondo's games [49,50] to be developed and will naturally, by extension, be the starting point for application in quantum games. In capital-dependent Parrondo's games, two losing games are played, in which game A is the toss of a single-biased coin with winning probability $p = \frac{1}{2} - \varepsilon$, while game B is played dependent on the total capital of the player. Coin c_1 with a winning probability of p_1 is tossed if the capital is divisible by M, otherwise c_2 is used with

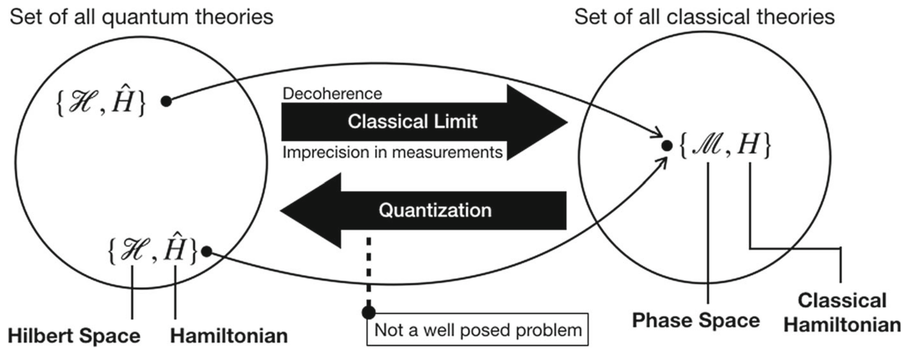

Fig. 1 Schematic summary of the connection between Quantum theories and Classical theories. Image adapted from *Advanced Quantum Theory, Lecture 1* by Tobias Osborne [43]

winning probability p_2. By choosing p_1 and p_2, we are able to show that playing games A and B individually over time gives a net loss, but playing a combination of games A and B results in a win. This is summarized in Fig. 2, where $p_1 = \frac{1}{10} - \varepsilon$, $p_2 = \frac{3}{4} - \varepsilon$ and $\varepsilon = 0.005$ [11].

The quantum equivalent of such games has been successfully simulated by various authors [51–55]. Meyer and Blumer devised a "coin"-flipping quantum model [51] by mapping the capital to the discrete one-dimensional position of a particle in Brownian motion. The quantum "coin" is represented by a two-level system such as a spin-$\frac{1}{2}$ particle in the superposition states $|+1\rangle$ and $|-1\rangle$, which are the eigenstates of the Pauli matrix σ_z. The position is $|x\rangle$, and the eigenstates are win or loss, respectively. When measured, the particle is in the state $|x, \alpha\rangle$, where the second component is the momentum of the particle. An initial state with zero capital and momentum is chosen, $\frac{1}{\sqrt{2}}(|0, +1\rangle + |0, -1\rangle)$. An unbiased "coin" is represented by the unitary matrix

$$U = \begin{pmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{pmatrix}. \tag{1}$$

Let $|x, \alpha\rangle$ denote the presence of a particle state in the Hilbert space, where $x \in \mathbb{Z}$ with momentum $\alpha \in \{\pm 1\}$, satisfying

$$\psi = \sum \psi_{x,\alpha}|x, \alpha\rangle, \quad \text{with } 0 \leq \psi_{x,\alpha} \in \mathbb{C} \quad \text{and} \sum |\psi_{x,\alpha}|^2 = 1, \tag{2}$$

so that $\psi_{x,\alpha}$ is the amplitude of the state $|x, \alpha\rangle$ and $|\psi_{x,\alpha}|^2$ is the probability that, if measured in this basis, the particle is observed to be in the state $|x, \alpha\rangle$. Thus, the quantum evolution according to the unitary operator given by Eq. 1 is

$$|x, \alpha\rangle \to \cos\theta|x, \alpha\rangle + i\sin\theta|x, -\alpha\rangle,$$

followed by

$$\cos\theta|x, \alpha\rangle + i\sin\theta|x, -\alpha\rangle \to \cos\theta|x+\alpha, \alpha\rangle + i\sin\theta|x-\alpha, -\alpha\rangle,$$

noting that including a potential with multiplication by an x-dependent phase $e^{-iV(x)}$, the evolution remains unitary. Thus, the quantum Parrondo's problem is reduced to picking parameters θ and $V(x)$ to achieve the paradoxical effect.

Without loss of generality, we have $\theta = \frac{\pi}{4}$ to maintain magnitudes. The quantum game is the evolution governed by the phase $e^{-iV(x)}$, for which games A and B, $V_A(x) = \frac{2\pi}{5000}x$ and $V_B(x) = \frac{\pi}{3}[1 - \frac{1}{2}(x \bmod 3)] + V_A(x)$. For these chosen potential functions and evolution phase, Meyer and Blumer managed to show the Parrondo's effect. Meyer has also successfully demonstrated that two fair "coins" can have the same winning effect by introducing noise to these quantum "coins", analogous to models of quantum ratchets [52].

While the game devised by Meyer and Blumer is able to quantize a capital-dependent Parrondo's game, the running time scales $O(2^n)$ with each game when simulated on a classical computer. The number of qubits scales linearly with each game. Therefore, Gawron and Miszczak implemented the quantum version of a capital-dependent Parrondo's game by using a $O(\log_2(n))$ qubits, for n number of Parrondo's games [53]. It is implemented similarly to classical games, except that the capital can only be checked with a quantum gate. In their implementation, finding a winning strategy, in a sequence of five games, is non-trivial and uncommon. Despite the difficulty, they managed to create a capital-driven scheme for quantum Parrondo's games using finitely limited number of qubits. The work by Košík, Miszczak and Bužek [54] is related to the work of Gawron and Miszczak. They employed the use of quantum random walks as a source of the

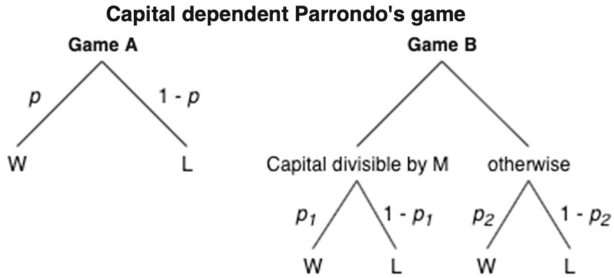

Fig. 2 Capital-dependent Parrondo's games involve tossing two biased coins [11]

randomness required in switching strategies in a quantum Parrondo's game. These strategies are described by completely positive (CP) maps. The payoff function is represented by a quantum observable on the tensor product of state spaces of all players. The model of quantum Parrondo's games is as follows: there are four registers (**C, D, X, O**), whose states are described by states in Hilbert spaces $\mathcal{H}_C, \mathcal{H}_D, \mathcal{H}_X$ and $\mathcal{H}_O$. The register **X** stores the capital; **D** is the coin register storing the strategy used; **C** is the chirality register which determines the strategy to use and **O** is the auxiliary register. The quantum game is defined on the Hilbert space, which takes the tensor product of all Hilbert spaces defined, $\mathcal{H} = \mathcal{H}_C \otimes \mathcal{H}_D \otimes \mathcal{H}_X \otimes \mathcal{H}_O$ such that:

(i) The Hilbert spaces $\mathcal{H}_j = \text{span}\{|k\rangle : k = 0, 1\}$ for $j \in \{C, D, O\}$. All operators on $\mathcal{H}_j$ is written in the basis $(|0\rangle, |1\rangle)$ and the Hilbert space $\mathcal{H}_X = \text{span}\{|x\rangle : x \in \mathbb{Z}\}$.

(ii) U is the unitary operator $\mathcal{H}_C$:

$$U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}. \tag{3}$$

(iii) The operator X is the NOT gate:

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{4}$$

(iv) The operator c-A is a controlled $SU(2)$ operator on $\mathcal{H}_C \otimes \mathcal{H}_D$, the operators c-$B_0$, c-$B_1$ are controlled $SU(2)$ operators on $\mathcal{H}_D \otimes \mathcal{H}_C \otimes \mathcal{H}_O$. In both cases, $\mathcal{H}_D$ is the target space. For any $SU(2)$ operator G, the parameterization

$$G(\theta, \alpha, \beta) = \begin{bmatrix} e^{i\alpha}\cos\frac{\theta}{2} & ie^{i\beta}\sin\frac{\theta}{2} \\ ie^{-i\beta}\sin\frac{\theta}{2} & e^{-i\alpha}\cos\frac{\theta}{2} \end{bmatrix}, \tag{5}$$

is used, with $\theta \in [0, \pi]$ and $\alpha, \beta \in [-\pi, \pi]$. We define

$$\text{c-}A = \pi_0 \otimes A + \pi_1 \otimes I; \tag{6}$$

$$\text{c-}B_j = \pi_0 \otimes (B_j \otimes \pi_0 + \pi_1 \otimes I) + \pi_1 \otimes I \otimes I, \tag{7}$$

for $j \in \{0, 1\}$.

(v) The gate MOD is the conditional operator:

$$MOD|x\rangle|o\rangle = \begin{cases} |x\rangle|o\rangle & \text{if } 3|x \\ |x\rangle|o \oplus 1\rangle & \text{otherwise.} \end{cases} \tag{8}$$

(vi) The operator S acting on $\mathcal{H}_C \otimes \mathcal{H}_X$ updates the capital register **X**

$$S = \pi_0 \otimes T_0 + \pi_1 \otimes T_1, \tag{9}$$

where $T_0|x\rangle = |x - 1\rangle$, $T_1|x\rangle = |x + 1\rangle$.

(vii) the gate MOD_{inv} acting on $\mathcal{H}_C \otimes \mathcal{H}_X \otimes \mathcal{H}_O$ is the conditional operator which resets the register **O**. If the state of the (**C,X**) register at the n-th step is (c_n, x_n), we have $x_{n-1} \equiv x_n - (2c_n - 1) \bmod 3$. At the n-th step, the operator MOD_{inv} flips $|o\rangle$ if and only if $x_{n-1} \equiv 0 \bmod 3$

The logical circuit shown in Fig. 3 can be simplified to obtain the circuit represented in Fig. 4. In this circuit, the operator W acting on $\mathcal{H}_D \otimes \mathcal{H}_C \otimes \mathcal{H}_X \otimes \mathcal{H}_O$ has the form

$$W = |0\rangle\langle 1|U \otimes \left[B_0 \otimes 1 \otimes |1\rangle\langle 0| + B_1 \otimes 1 \otimes |0\rangle\langle 1|\right] + |0\rangle\langle 1|U \otimes A \otimes 1 \otimes X. \tag{10}$$

The operators S and MOD_{inv} depend non-trivially only on $|c\rangle$. In the simplified diagram in Fig. 4, the notation $\mathcal{H}_W \equiv \mathcal{H}_D \otimes \mathcal{H}_C$ and further express the state of the whole system using the eigenvectors of the translation operator on $\mathcal{H}_X$:

$$|\phi_k^j\rangle = \sum_{x \in \mathbb{Z};\, x \equiv j \bmod 3} e^{ikx}|x\rangle, \tag{11}$$

for $j \in \{0, 1, 2\}$, $k \in [-\pi, \pi]$. It is clear that $T_0|\phi_k^j\rangle = e^{ik}|\phi_k^{j\ominus 1}\rangle$ and $T_1|\phi_k^j\rangle = e^{-ik}|\phi_k^{j\oplus 1}\rangle$, where we set

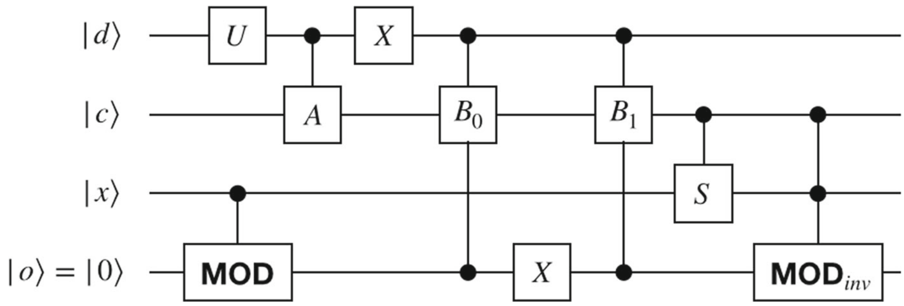

Fig. 3 The quantum circuit diagram for the quantum Parrondo's game. Image adapted from Ref. [54]

$|\phi_k\rangle = \sum_{j=0}^{2} |\phi_k^j\rangle$. The inverse transformation is given by the expression

$$|x\rangle = \int_{-\pi}^{\pi} \frac{\mathrm{d}k}{2\pi} e^{-ikx} |\phi_k\rangle. \tag{12}$$

The action of $W{\cdot}$MOD on the state $|\chi\rangle|\phi_k^j\rangle|0\rangle$ gives

$$|\chi\rangle|\phi_k^0\rangle|0\rangle \rightarrow \Big(\pi_{01} U \otimes B_0 \otimes 1 \otimes \pi_{10} + \pi_{10} U \otimes A \otimes 1 \otimes X\Big)|\chi\rangle|\phi_k^0\rangle|0\rangle; \tag{13}$$

$$|\chi\rangle|\phi_k^{1,2}\rangle|0\rangle \rightarrow \Big(\pi_{01} U \otimes B_1 \otimes 1 \otimes \pi_{01} + \pi_{10} U \otimes A \otimes 1 \otimes X\Big)|\chi\rangle|\phi_k^{1,2}\rangle|1\rangle, \tag{14}$$

where $\pi_{ab} \equiv |a\rangle\langle b|$. Application of the operator S and resetting the last register with $\mathrm{MOD_{inv}}$ gives the evolution operator

$$E|\chi\rangle|\phi_k\rangle|0\rangle = \Big\{(M_{10} + M_{11})|\chi\rangle|\phi_k^0\rangle + (M_{10} + M_{01})|\chi\rangle|\phi_k^1\rangle + (M_{11} + M_{00})|\chi\rangle|\phi_k^2\rangle\Big\}|0\rangle, \tag{15}$$

with

$$M_{jd} = e^{s_d ik}(\pi_{01} U \otimes \pi_d B_j + \pi_{10} U \otimes \pi_d A), \tag{16}$$

where $j, d \in \{0, 1\}$, $s_d = 1 - 2d$. Multiple application of E on the initial state gives

$$E^n|\chi\rangle|\phi_k\rangle|0\rangle = \Big(\mu_0^{(n)}|\chi\rangle|\phi_k^0\rangle + \mu_1^{(n)}|\chi\rangle|\phi_k^1\rangle + \mu_2^{(n)}|\chi\rangle|\phi_k^2\rangle\Big)|0\rangle. \tag{17}$$

The terms $M_j^{(n)}$ are related by the matrix equation

$$\begin{bmatrix} \mu_0^{(n+1)} \\ \mu_1^{(n+1)} \\ \mu_2^{(n+1)} \end{bmatrix} = \begin{bmatrix} 0 & M_{10} & M_{11} \\ M_{01} & 0 & M_{10} \\ M_{00} & M_{11} & 0 \end{bmatrix} \begin{bmatrix} \mu_0^{(n)} \\ \mu_1^{(n)} \\ \mu_2^{(n)} \end{bmatrix}, \tag{18}$$

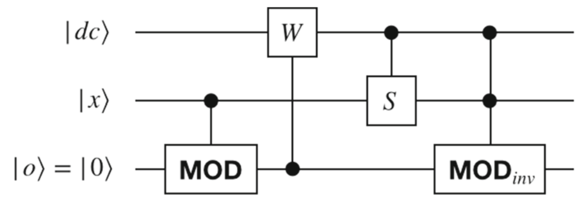

Fig. 4 A simplified quantum circuit, with operator W for the quantum Parrondo's game. Image adapted from Ref. [54]

with $\mu_j^{(0)} = 1$. The problem can be solved by computing the eigensystem of this 12×12 matrix, which can be solved with a running time of around $O(n^3)$. Hence, the work by [53] and [54] improves on the early work of [51], both in the number of qubits needed and in running time.

Flitney also studied a one-dimension quantum walk by mixing two different "coin" operators (both with negative expectation) to produce a positive-biased walk [55]. This is achieved by introducing a phase factor term into the "coin" operator, with different phases giving games A and B. From the construction of the general initial state for the quantum walk by Tregenna et al. [56],

$$|\psi(x, 0)\rangle = \big(\sqrt{\eta}|R\rangle + e^{i\mu}\sqrt{1-\eta}|L\rangle\big) \otimes |0\rangle, \tag{19}$$

with a "coin" operator given by the unitary matrix

$$U(\rho, \theta, \phi) = \begin{pmatrix} \sqrt{\rho} & e^{i\theta}\sqrt{1-\rho} \\ e^{i\phi}\sqrt{1-\rho} & -e^{i(\theta+\phi)}\sqrt{\rho} \end{pmatrix}. \tag{20}$$

Flitney constructs the Parrondo's games by choosing an initial state $|\psi_0\rangle = (|0, L\rangle - |0, R\rangle)/\sqrt{2}$, and a symmetric operator,

$$U(\rho, \alpha) = \begin{pmatrix} e^{i\alpha}\sqrt{\phi} & i\sqrt{1-\rho} \\ i\sqrt{1-\rho} & e^{-i\alpha}\sqrt{\phi} \end{pmatrix}, \tag{21}$$

achieved by multiplying the operator in Eq. 20 by a phase $e^{i\alpha}$, and setting $\theta = \phi = \frac{\pi}{2} - \alpha$. With $\rho = \frac{1}{2}$ and $\alpha = 0$, an unbiased quantum walk is obtained.

However, with $\rho = \frac{1}{2}$ a bias can be introduced by the use of a non-zero phase factor α. If we restrict $\alpha \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, then $\alpha > 0$ causes average motion to the left. By introducing the operators

$$A(\alpha) = U(\frac{1}{2}, \alpha), \quad B(\beta) = U(\frac{1}{2}, \beta), \tag{22}$$

for some α,$\beta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Games A and B can now be selected by choosing α,$\beta > 0$, giving both games a net negative bias. The alternating play of A then B n-times from the initial state $|\psi_i\rangle$ produces the final state $|\psi_f\rangle$ given by

$$|\psi_f\rangle = (B \otimes A)^n |\psi_i\rangle. \tag{23}$$

All repeated sequences of four games were tested, of which AB, ABB and $ABBB$ were found to give Parrondo-like effect after 100 time steps. In each case, for a sufficiently large number of time steps, all sequences eventually show a return to a negative expectation value. The domain of α and β as well as the range of time steps that gives positive expectation can be found in Ref. [55]. However, it should be noted that the paradox disappears for quantum walk approach, in the infinite time or asymptotic limit.

2.2 History-dependent Parrondo's games

The seminal history-dependent games build on capital-dependent games, with game B independent of capital—we call it game B' so as not to confuse with the earlier game B. In game B', there are four biased coins and the coin to be played depends on the *history* of the outcomes from previous games [57,58]. The dynamics of the game is described in Fig. 5. Playing games A and B' individually will produce a net loss over time, but playing a combination of games A and B' results in an overall win. As an illustration, the history-dependent Parrondo's games use biased dice with winning probabilities $p_1 = \frac{9}{10} - \varepsilon$, $p_2 = p_3 = \frac{1}{4} - \varepsilon$ and $p_4 = \frac{7}{10} - \varepsilon$, with $\varepsilon = 0.005$.

Quantum analogies of history-dependent games have been realized by various groups [59–62]. Flitney et al. [59] have demonstrated a quantum history-dependent game by considering the implementation of $SU(2)$ operation on a qubit, where game B consists of four $SU(2)$ operations controlled by the result of the previous two games. The $SU(2)$ operator acting on a qubit can be written as

$$A(\theta, \gamma, \delta) = P(\gamma) R(\theta) P(\delta) = \begin{pmatrix} e^{-i(\gamma+\delta)/2} \cos\theta & -e^{-i(\gamma-\delta)/2} \sin\theta \\ e^{i(\gamma-\delta)/2} \sin\theta & e^{i(\gamma+\delta)/2} \cos\theta \end{pmatrix}, \tag{24}$$

where $\theta \in [-\pi, \pi]$ and γ,$\delta \in [0, 2\pi]$. This is game A, achieved physically on a polarized photon by rotating the plane of polarization by θ (R) between two birefringent media (P) that results in phase differences of γ and $delta$, respectively, between the horizontal and vertical planes of polarization. Game B is constructed using four $SU(2)$ operations, in the form of Eq. 24:

$$B(\phi_i, \alpha_i, \beta_i) = \begin{bmatrix} A(\phi_1, \alpha_1, \beta_1) & 0 & 0 & 0 \\ 0 & A(\phi_2, \alpha_2, \beta_2) & 0 & 0 \\ 0 & 0 & A(\phi_3, \alpha_3, \beta_3) & 0 \\ 0 & 0 & 0 & A(\phi_4, \alpha_4, \beta_4) \end{bmatrix}, \tag{25}$$

for $i \in \{1, 2, 3, 4\}$. This acts on the state $|\psi_{t-2}\rangle \otimes |\psi_{t-1}\rangle \otimes |i\rangle$, where $|\psi_{t-2}\rangle$ and $|\psi_{t-1}\rangle$ represent the

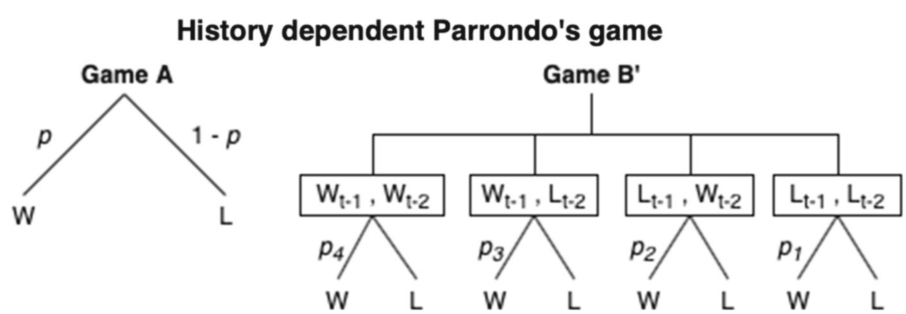

Fig. 5 Construction of the history-dependent games, where game B' has four possible historic outcomes {LL, LW, WL, WW}. W_j and L_j denote a "win" and "loss" at time step j, respectively. Depending on the four historic outcomes of the game, the probability of winning is p_i, $i = \{1, 2, 3, 4\}$ [57]

results of the previous two games and $|i\rangle$ is the initial state of the target qubit. That is,

$$B|q_1q_2q_3\rangle = |q_1q_2b\rangle, \tag{26}$$

where $q_1, q_2, q_3 \in \{0, 1\}$ and b is the output of game B. n successive games of B can be computed by

$$\begin{aligned}|\psi_f\rangle =&(I^{\otimes n-1} \otimes B)(I^{\otimes n-2} \\ &\otimes B \otimes I)(I^{\otimes n-3} \otimes B \otimes I^{\otimes 2}) \\ &\ldots (I \otimes B \otimes I^{\otimes n-2})(B \otimes I^{\otimes 1})|\psi_i\rangle,\end{aligned} \tag{27}$$

with $|\psi_i\rangle$ being the initial state of $n+2$ qubits. The first two qubits of $|\psi_i\rangle$ are left unchanged and only used as the input for the first game of B. I is the identity operator for a single qubit. The result of other game sequences can be computed in a similar manner. The payoff for such games is defined by the authors as $+1$ for state $|1\rangle$ and -1 for state $|0\rangle$. Then the expected payoff from a sequence of games for the final state $|\psi_f\rangle$ is

$$\langle \$ \rangle = \sum_{j=0}^{n} \left((2j - n) \sum_{j'} \left| \langle \psi_j^{j'} | \psi_f \rangle \right|^2 \right), \tag{28}$$

where the second summation is taken over all basis states $\langle \psi_j^{j'}|$ with j 1's and $n - j$ 0's. An interesting case developed by Flitney et al. (2000) is the case of the game sequence AAB, for which they were able to show that the classical result of a history-dependent game is a subset of the possible quantum results.

Advancing their work on quantum versions of history-dependent games, Flitney et al. (2004) demonstrated that the history-dependent games can also be represented as position expectation in quantum walks [60]. Their work uses multiple quantum coins, but they are able to show that it is analogous to the classical result when a single quantum coin is used. Importantly, as previously noted, they have shown that the results from classical history-dependent Parrondo's games are a subset of possible quantum history-dependent Parrondo's games.

Khan et al. built on the work of Flitney et al. (2004) by studying history-dependent quantum Parrondo's games under the effect of decoherence being played in different sequences for amplitude damping (AD), depolarizing (D) and phase damping (PD) channels [61]. They noted that the payoffs are enhanced in the presence of decoherence for the maximally entangled state in the case of a single game of the sequence AAB for AD channel. This is due to the decoherence causing constructive interference of quantum phases that leads to an enhanced payoff. In contrast, in the case of the D and PD channels, phases undergo destructive interference resulting in a decrease in payoff, a losing game sequence. Losing game A, under repetition, has payoffs influenced only by the AD channel and remains a losing game. The losing game B, and BB has similar dynamics for AD and D channels. Lastly, it was observed that for any sequence when played in succession, the PD channel does not influence the dynamics of the outcome.

While Bleiler and Khan appreciate the scientific merits of the results by Flitney et al. (2004), the quantization is non-Markovian as it does not result in a stable state solution and therefore, does not carry any game theoretic significance [62]. Thus, Bleiler and Khan introduced their own method of quantization—Bleiler formalism [63]. Under proper quantization according to the Bleiler formalism, they were able to derive the same results and show that classical results form the subset of quantum games under special cases.

2.3 Cooperative Parrondo's games

Classical cooperative Parrondo's games involve multi-agents. Such games consider an ensemble of agents and use "social" rules in which the probabilities of the games are defined in terms of the actual state of the neighbours of a given player [64–68]. The seminal cooperative Parrondo's game builds on the capital-dependent Parrondo's games. Consider an ensemble of agents arranged in a circle. In each iteration of the game, a random agent i is chosen, where either game A or B" is played. Game A is as defined in the capital Parrondo's game. Game B" can be played using four biased coins depending on the *capital* of the neighbours of the randomly chosen agent. The dynamics of the game is described in Fig. 6.

A win in this case is defined as a positive mean capital of all agents. A well-known example utilizing 30-agent cooperative Parrondo's game uses biased dice with winning probabilities $p_1 = 1.0$, $p_2 = p_3 = 0.16$ and $p_4 = 0.7$ [64].

Quantum cooperative games follow the same dynamics where the "social rules" are governed by the state of a qubit's "neighbours" [69,70]. Bulger, Freckleton and Tawmley demonstrated the quantum analogue of

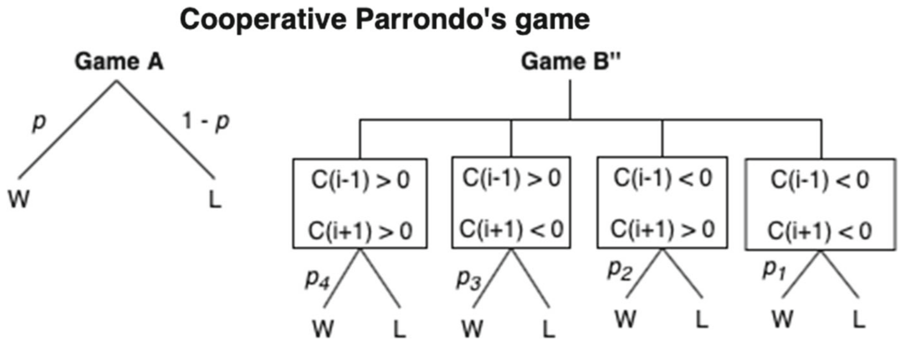

Fig. 6 Construction of the cooperative Parrondo's game, where game B" has four possible cooperative outcomes depending on the capital of agent $i-1$ and $i+1$, where i is a randomly chosen agent in each iteration of the game. The function $C(j)$ is the last change in capital of agent j. Depending on the four cooperative outcomes of the game, the probability of winning is p_i, $i = \{1, 2, 3, 4\}$ [64]

cooperative Parrondo's effect through numerical simulations on a position expectation quantum walk [69]. By extending the work of Košík et al. [54], Bulger has allowed the quantum system to evolve by performing successive unitary operators and demonstrated the quantum analogue of [64]. The quantum walk evolves on a quantum system comprising registers that represent the selected agent, or in the words of the authors, which participant is currently in control of the particle (called the *participant selection* subsystem), which game is being played (the *subwalk selection* subsystem), the state of each agent (the participant's *chirality* bit) and the particle's position (the *position* subsystem). A subwalk and participant are chosen at the beginning of each iteration by operating a fixed unitary operator to both the subwalk selection and participant selection subsystem. Then, the chosen participant's chirality qubit is rotated through an angle depending on the states of the participant's neighbours' chirality qubits, and on the selected subwalk. Lastly, the particle is advanced forward or backward according to the chirality qubit of that participant. We reproduce an illustration [69] to visualize this, as shown in Fig. 7.

Although Fig. 7 masks some details, it illustrates the procedure that takes place in each iteration. In each iteration of cooperative walk, the wheel is rotated to select a participant, updates the participant's chirality depending on the subwalk chosen, then the wheel is moved along the axis in the direction indicated by the chirality of the participant.

The walk forms a unitary operation on the space

$$\mathcal{H}_S \otimes \mathcal{H}_W \otimes \mathcal{H}_{X_1} \otimes \ldots \mathcal{H}_{X_p} \otimes \mathcal{H}_C, \tag{29}$$

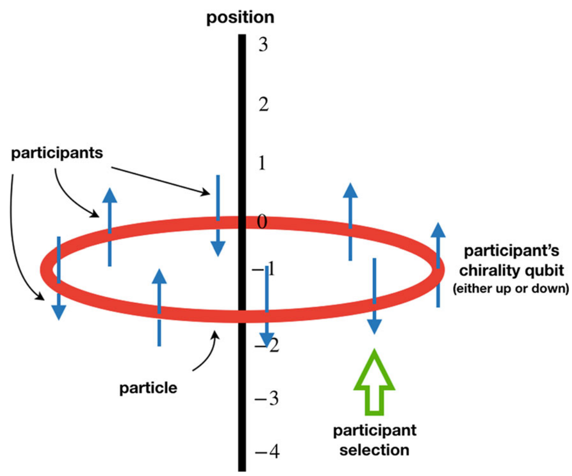

Fig. 7 The participants are placed on a wheel which moves vertically on an axis representing the position of the walk. The state of each participant is represented by an arrow, pointing up for a chirality of $|1\rangle$. The participant selection state is shown by the rotation of the wheel about its centre so that the participant above the selection is chosen to participate in the subwalk. Image adapted from Ref. [69] under CC BY-NC-SA 3.0

where

- $\mathcal{H}_S = \text{span}\{|0\rangle, \ldots, |p-1\rangle\}$ determines which participant moves the particle in the current iteration,
- $\mathcal{H}_W = \text{span}\{|0\rangle, \ldots, |r-1\rangle\}$ determines which subwalk is used (in the paper, $r = 2$),
- $\mathcal{H}_{X_j} = \text{span}\{|0\rangle, |1\rangle\}$ records the most recent move of the particle, backward and forward, respectively, and

- $\mathcal{H}_C = \text{span}\{\ldots |-1\rangle, |-0\rangle, |1\rangle \ldots\}$ determines which participant moves the particle in the current iteration.

One iteration of the walk, denoted by $C(U_w, U_s, \Theta)$ can be expressed as

$$C(U_w, U_s, \Theta) = \sum_{j=0}^{p-1} \sum_{k=0}^{r-1} \pi_j^{(S)} \pi_k^{(W)} \left(\pi_0^{(X_j)} T_-^{(C)} + \pi_1^{(X_j)} T_+^{(C)}\right) A_{jk} U_w^{(W)} U_s^{(S)}, \tag{30}$$

where $\pi_j^{(S)}$,$\pi_k^{(W)}$ and $\pi_0^{(X_j)}$ are projections acting on the subsystem $\mathcal{H}_S$, $\mathcal{H}_W$ and $\mathcal{H}_{X_j}$, respectively, $T_\pm^{(C)}$ is the increment/ decrement operator acting on subsystem $\mathcal{H}_C$, and

$$A_{jk} = \pi_0^{(X_{j-1})} G_{0k}^{(X_j)} \pi_0^{(X_{j+1})} + \pi_0^{(X_{j-1})} G_{1k}^{(X_j)} \pi_1^{(X_{j+1})} + \pi_1^{(X_{j-1})} G_{2k}^{(X_j)} \pi_0^{(X_{j+1})} + \pi_1^{(X_{j-1})} G_{3k}^{(X_j)} \pi_1^{(X_{j+1})} \tag{31}$$

with G_{sk} taking the form

$$G_{sk} = \begin{pmatrix} \cos \frac{\Theta_{sk}}{2} & i \sin \frac{\Theta_{sk}}{2} \\ i \sin \frac{\Theta_{sk}}{2} & \cos \frac{\Theta_{sk}}{2} \end{pmatrix}, \tag{32}$$

and Θ_{jk} is a $4 \times r$ real matrix, whose entry specifies the rotation angle. The number of rows of Θ is fixed at four because there are only four possible combinations of the states of the selected participant's two neighbours. With this formalism, the authors were able to show that the combination of two "losing" subwalks gives a "winning" subwalk—Parrondo's effect [69].

A large part of quantum information theory is the initial input state and how it affects the final outcome. Pawela and Sładkowski wanted to investigate whether the initial quantum state affects the final outcome in quantum Parrondo's games. With this motivation, they modelled cooperative Parrondo's paradox using multi-dimensional quantum random walks with biased coins. The initial states of the coins were chosen to be

1. Greenberger–Horne–Zeilinger state (GHZ state), $|C\rangle = \frac{1}{\sqrt{2}}(|LLL\rangle + |RRR\rangle)$,
2. the W state, $|C\rangle = \frac{1}{\sqrt{3}}(|LLR\rangle + |LRL\rangle + |RLL\rangle)$,
3. a separable state, $|C\rangle = \frac{1}{2\sqrt{2}}(|L\rangle - |R\rangle)^{\otimes 3}$, and
4. a semi-entangled state $|C\rangle = J|LLL\rangle$, where $J(\omega) = \exp\left(i\frac{\omega}{2}\sigma_x^{\otimes 3}\right) = I^{\otimes 3} \cos \frac{\omega}{2} + i\sigma_x^{\otimes 3} \sin \frac{\omega}{2}$, where $\omega \in [0, \pi/2]$ is a measure of entanglement.

They concluded that if the initial state of the coins is separable and one game is a winning one, then the combination of games A and B can become a losing game. Such an effect is not seen when the initial state of the coins is set to be the GHZ state, where both games A+B and [2,2] are always non-losing games. This shows that the choice of the initial state may be crucial for the paradoxical behaviour. This work provides an alternative strategy to easily distinguish between the GHZ and W states—a useful contribution to quantum information theory that is derived from quantum Parrondo's paradox.

3 Implications of quantum Parrondo's games

While classical Parrondo's games can be realized by performing coin tosses, quantum Parrondo's games can only be "played" via thought experiments and proven analytically and computationally. The results arising from some quantum Parrondo's games have been computationally proven. Fundamentally, quantum game theoretic experiments have been thought up, but not realized. Si [71] has proposed a method of implementing quantum Parrondo's games by a beam-splitter as the "coin" and each circularly polarized photon fired at a beam-splitter as an iteration of a game. There are two possible outcomes of the game, "win" and "'loss"—a photon passing through a beam-splitter and scattering off a beam-splitter, respectively. To the best of our knowledge, this experiment has not yet been performed. If realized, this is likely the first experimental representation for implementing a quantum Parrondo's game.

While the theory of quantum Parrondo's games has been developed extensively over the past decades, there have only been few applications beyond the theory of quantum random walks [72–75], all of which are built upon the capital-dependent games. Applications of quantum Parrondo's games also include quantum Markov processes via construction of unitary matrices using operations. In particular, Li et al. [76] use $SU(2)$ operators and Chen et al. [77] use position-space quantization operators. Several studies discussed in the previous sections have revealed that the classical result from Parrondo's games form a subset of quantum results. Therefore, the study of quantum Parrondo's games can possibly reveal results in classical Parrondo's games that may not be easily proven or explained using classical means. Quantum versions of Parrondo's games have an advantage over games restricted to classical strategies—development in quantum Parrondo's games can be used to explain critical phenomena in classical games. By extension, the use of non-commutative algebra extends the Hilbert space and therefore is a proper quantum Parrondo's game, which has an advantage over classical games. These include revealing deeper applications to the fields of biology and ecology, computational economics, physical chemistry, chaos and nonlinear dynamics and engineering. In particular, developments in economics have seen quantum game theoretic applications [78] and have been at the core of classical Parrondo's paradox research.

While the extent of current research focuses on simulating quantum analogies of classical Parrondo's games [79,80], the developments in quantum Parrondo's paradox mainly come from translating and simulating classical examples. The focus of quantum Parrondo's games has largely been developed through commutative algebra. This leaves a gap in the full understanding of quantum game theory—Commutative algebraic representations of classical action is incomplete and advances in quantum computing can also be motivated by exploring non-commutative algebra, which in turn will open up developments in quantum Parrondo's games.

Furthermore, quantum versions of Parrondo's games have been used to discover phenomena that appear in quantum versions of Parrondo's games but are hard to prove in classical games. Examples include the research by Zhu et al. [81] and Wang et al. [82]. They further developed classical versions of the capital-dependent Parrondo's games by considering the case for $M = 4$ and for any general M, respectively. They have shown that in the case $M = 4$, there is no definite stationary probability distribution that the final capital depends on. The final capital will depend on the parity of the initial start state. This means that the ordered process of playing games A and B gives an undetermined final result, while playing games A and B at random produces a deterministic final result. Extending this to the general case for M, they have found that the process of the game is purely determined by the parity of the initial state chosen and proven by considering the discrete-time Markov chain of the Parrondo's game.

Additionally, the development of quantum computation and quantum game theoretic concepts [83–85] has seen Parrondo's paradox being used to explain aspects of quantum information theory such as quantum coherence and the effects of correlated noise and decoherence [86,87]. In particular, Lee et al. have devised a model in which the alternation of two systems of decoherence dynamics can decrease the decoherence rate of each separate dynamics [88]. Aspects of quantum decoherence and quantum noise continues to be an emerging field in Physics, which quantum game theory can shed light on. Advancing the research in quantum computation also gives rise to developments in the application of Parrondo's paradox in quantum game theory [89]. Other areas of research, not found in classical Parrondo's games, also include the study of entangled states in quantum Parrondo's games [70,90,91].

Further implementations can also be linked to develop the field of chaotic quantum game theory. Consider the standard classical Parrondo's games "Game A + Game B = Game C", there are two broad classes of problems in the study of chaotic Parrondo's games.

(i) Parrondo's games with chaotic switching [92,93], here, the chaos is in the "+".
(ii) Chaotic systems combining to give order [94–97], here, games A and B are chaotic, and the outcome, game C is order.

An example of chaos and nonlinear dynamics being used in class (i) problems is the inclusion of chaotic maps as a means of determining the switching mechanism between games A and B as described in capital-dependent games. The random switching is replaced by a recursive relation. In [92], one of the chaotic map used was the logistic map:

$$x_{n+1} = ax_n(1 - x_n). \tag{33}$$

At each round of Parrondo's games (round n), a value from the chaotic sequence x_n is compared with γ. If $x_n \le \gamma$, game A will be played, but if $x_n > \gamma$, game B will be played. For a random switching strategy, γ is equivalent to the proportion of game A played after n discrete-time steps. However, this is not necessarily true for a chaotic switching strategy. γ is determined by observing the normalized sequence values of x_n between 0 and 1. γ is chosen for which the Parrondo's effect is observed for a fixed value of a. This is repeated for other recursive chaotic maps in one-dimension (e.g. Tent map, Sinusoidal map and Gaussian map) and two-dimensions (e.g. Henon map and Lozi map).

We note that in classical Parrondo's paradox, noise is necessary to allow the paradoxical result to emerge.

However, in the study of quantum Parrondo's games, noise can also appear in the classical limit (see Fig. 1) of quantum systems. The noise necessary to observe paradoxical results and the noise that results from the classical limit are fundamentally different. This provides two entry points for quantum chaotic and dynamical systems to be introduced to Parrondo's games to further develop the field of quantum Parrondo's games.

Class (ii) has been widely applied to explain phenomena in chaotic and dynamical systems in physics and chaos control [12,21,22,98,99]. The theoretical framework and derivations leading to Parrondo's paradox are system dependent. Acknowledging that unpredictability is inherently built into non-commutative and non-unitary quantum systems, class (ii) problems can be further explored by considering quantum systems as the chaotic games giving rise to deterministic results.

Quantum versions of class (i) problems have yet to be fully explored. We have assessed that research in this area can be further expanded theoretically, contributing to a deeper understanding of stochastic chaotic and dynamical systems. For example, these can be applied to areas of statistical mechanics where noise is a source of non-equilibrium switching between equilibrium dynamics, thereby giving insights on order and complexity.

4 Conclusion

In this review paper, we have discussed the development of Parrondo's paradox from classical game theoretic definitions to quantum versions of these games and proposed extension to non-commutative quantum formalism of quantum Parrondo's games as well as applications to chaotic quantum Parrondo's games. While the current development of quantum Parrondo's games is largely determined by the advances from classical games, we wish to point out that the main advantage comes when the independent pursuit of research quantum games gives rise to results in the classical world that are previously unexplained.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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